ALGEBRA PRELIMINARY EXAMINATION

JANUARY 2004

NOTES. $\mathbb{Z}$ and $\mathbb{Q}$ are the integers and the rational numbers respectively. All rings are commutative with identity unless specifically indicated otherwise.

(1) Prove that if n and m are relatively prime integers and G is an abelian group of order nm, then G can be decomposed into a direct product of a group of order n and a group of order m.

(2) Show that there is no simple group of order 132.

(3) Show that A_6 has no subgroup of order 90.

(4) Show that the group of units in an imaginary quadratic field is finite. (This terminology means the ring

$$R = \begin{cases} \mathbb{Z}[\sqrt{d}], & \text{if } d \equiv 2, 3 \bmod(4) \\ \mathbb{Z}[\frac{1+\sqrt{d}}{2}], & \text{if } d \equiv 1 \bmod(4). \end{cases}$$

where $d < 0$.) Note: for this problem we will be happy if you show the unit group is finite for only one of the above cases (pick your favorite one).

(5) Let R be a Noetherian ring and T an extension of R that is finitely generated over R as a ring. Show that T is Noetherian.

(6) A ring R (commutative with identity) is said to be zero-dimensional if every prime ideal is maximal. Show that any zero-dimensional domain is a field and that any finite commutative ring with identity is zero-dimensional.

(7) Recall that an $R-$module, P is said to be projective if given an $R-$module epimorphism $f : A \longrightarrow B$ and an $R-$module homomorphism $\phi : P \longrightarrow B$ then there is an $R-$module homomorphism $\overline{\phi} : P \longrightarrow A$ such that $f \cdot \overline{\phi} = \phi$.

$$\begin{array}{ccccc} & & P & & \\ & \overline{\phi} \swarrow & \downarrow \phi & & \\ A & \xrightarrow{f} & B & \longrightarrow & 0 \end{array}$$

Show that P is projective if and only if every short exact sequence of the form $0 \longrightarrow A \longrightarrow B \longrightarrow P \longrightarrow 0$ is split exact.

(8) Show that, as a polynomial over $\mathbb{Q}[x, y]$, $f(x, y) = x^4 + 3y + x^3y^2 + 2xy^2$ is irreducible.

(9) Let R be a commutative ring with identity of characteristic 0. Suppose that $I \subseteq R$ is an ideal that is maximal with respect to the property that R/I is of characteristic 0. Show that I is prime.

(10) Find all irreducible polynomials of degree 3 over $\mathbb{Z}_2$. Compute the Galois groups over $\mathbb{Z}_2$: $\text{Gal}(\mathbb{Z}_2[x]/(f(x)), \mathbb{Z}_2)$ where $f(x)$ are the degree 3 irreducible polynomials over $\mathbb{Z}_2$.