ALGEBRA PRELIMINARY EXAMINATION

SUMMER 1997

Notes. $\mathbb{Z}$ and $\mathbb{Q}$ refer to the integers and the rational numbers respectively. All rings are commutative with identity unless specifically noted otherwise. The word “domain” means “integral domain,” and “PID” means ”principal ideal domain.”

(1) Show that any group of order p^2 (with p a prime) is abelian.

(2) Find (up to isomorphism) all groups of order 8.

(3) Find all abelian groups of order 100.

(4) Let R be a commutative ring with identity, and I an ideal maximal with respect to being non-principal (such ideals exist in every non-PID). Show that I is prime.

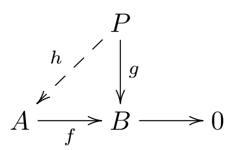
(5) Show that the polynomial $f(x) = x^4 + 2x^2 + 4x - 2$ is irreducible in $\mathbb{Q}[x]$. What can you say about the ring $\mathbb{Q}[x]/(f(x))$?

(6) A commutative ring, R, with quotient field K is said to be *integrally closed* if given any $z \in K$ such that z is the root of a *monic* polynomial in $R[x]$, then z is in R. Show that any UFD is integrally closed.

(7) Let $\mathbb{F}$ be a field with $char(\mathbb{F}) \neq 2$. Let $\mathbb{K}$ be an extension of $\mathbb{F}$ of degree 2. Show that $\mathbb{K}$ is Galois over $\mathbb{F}$.

(8) Let $f(x) = x^5 - 1$. Compute $Gal(f, \mathbb{Q})$ and $Gal(f, \mathbb{F}_5)$.

(9) We say that the $R-$module P is *projective* if given any $R-$module surjection $f : A \longrightarrow B$ and an $R-$module homomorphism $g : P \longrightarrow B$, there is an $R-$module homomorphism $h : P \longrightarrow A$ such that $fh = g$ (as per the diagram below)



Show that P is projective if and only if there is a free $R-$module F such that $F \cong P \oplus K$ for some $R-$module K.

(10) Let R be a commutative ring with identity, A, B, and C $R-$modules and $f : A \longrightarrow B$ an $R-$module homomorphism. Show that there is an induced $R-$module homomorphism $f \otimes 1_C : A \otimes_R C \longrightarrow B \otimes_R C$. Also show that if f is onto, then so is $f \otimes 1_C$.