Instructions. Answer **any 4 short questions, and any 4 long questions.** Clearly mark which questions you wish to be graded on this sheet, or else 1-4 and 6-9 will be graded. Show all work, and explain your answers clearly. Solutions will be graded on correctness and clarity. All answers should include some explanation.

Shorter questions: (5 points each)

1. How many multisets are subsets of the multiset $\{1,1,2,2,2,3,3,3,4,4,4,4\}$?
2. How many permutations on [9] have cycles of length 3, 2, 2, and 2?
3. How many monomials of degree 7 are there in the variables x_1, x_2, x_3, x_4, and x_5?
4. A graph on 4 nodes has the following adjacency matrix:

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

   How many paths of length 20 start and end at node 3? Your answer must be a number.

5. Let N be a collection of n indistinguishable objects, and let X be a collection of x distinguishable objects. How many injective functions $f : N \to X$ are there?

Longer questions: (10 points each)

6. Use generating functions to solve the following recurrence relation:

$$a_n = 4a_{n-1} + 5a_{n-2} \text{ for } n \geq 2, a_0 = 0, a_1 = 1$$

   Your final answer should be an explicit formula for a_n in terms of n.

7. Use the principle of inclusion exclusion to count the number of pairs of functions (f, g) where $f, g : [m] \to [n]$ and for each $\ell \in [n]$, there is a $k \in [m]$ such that $f(k) = \ell$ or $g(k) = \ell$. That is, $\text{image}(f) \cup \text{image}(g) = [n]$. In your answer, state the underlying set and the properties you used to apply inclusion exclusion.
8. Use generating functions to prove that the number of partitions of n where each part occurs at most k times is equal to the number of partitions of n where each part is not divisible by $k+1$. In your proof, explain how you derived your generating functions.
9. Give a combinatorial proof of the following identity:

$$\sum_{k=1}^{n} k\binom{n-1}{k-1} = (n-1)2^{n-2} + 2^{n-1}$$

10. How many pairs of lattice paths (A, B) are there where path A starts at $(0,0)$ and ends at $(6,8)$, path B starts at $(2,0)$ and ends at $(10,6)$, and A and B never intersect?