Algebra Preliminary Examination
September 2008

Instructions: Begin each question on a new sheet of paper.
In this exam, all rings have identity and all modules are unital.

1. For each finite abelian group G and each integer $n \geqslant 0$, let $f_G(n)$ denote the number of elements in G of order n. For finite abelian groups G and H, prove that $G \cong H$ if and only if $f_G = f_H$.
2. Let p be a prime number and let G be a finite p-group. For each positive integer d such that $d \big| |G|$, prove that G has a normal subgroup H such that $|H| = d$.
3. Let R be a commutative ring with identity. Let I and J be ideals of R such that $I + J = R$. Prove that $I \cap J = IJ$ and that $R/IJ \cong R/I \times R/J$.
4. Recall that the Jacobson radical of a commutative ring is the intersection of its maximal ideals. Let n be an integer such that $n \geqslant 2$. Compute the Jacobson radical of $\mathbb{Z}/(n)$.
5. (a) Define the terms "Euclidean domain" and "principal ideal domain".
   (b) Prove that every Euclidean domain is a principal ideal domain.
6. Let $K \subseteq L$ be an extension of finite fields. Prove that this extension is Galois and has cyclic Galois group.
7. Let K be a field and let $K^{\times}$ denote the multiplicative group of nonzero elements of K. Prove that every finite subgroup of K is cyclic.
8. Let R be a ring and consider an exact sequence of R-modules

$$0 \to L \to M \to N \to 0.$$

   (a) Prove that, if N is projective, then M is projective if and only if L is projective.
   (b) Provide an example where L and M are projective but N is not projective. Justify your response.
9. Let R be a ring with identity and let M be an R-module. Recall that an R-module $N \neq 0$ is *simple* if its only submodules of N are 0 and N.
   (a) Prove that, if M is simple, then $\mathrm{Hom}_R(M, M)$ is a division ring.
   (b) Find an example where $\mathrm{Hom}_R(M, M)$ is not a division ring. Justify your response.
10. Let R be a ring and consider the following commutative diagram of left R-module homomorphisms with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M_1 & \xrightarrow{f_1} & M_2 & \xrightarrow{f_2} & M_3 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle h_1} & & \downarrow{\scriptstyle h_2} & & \downarrow{\scriptstyle h_3} & & \\ 0 & \longrightarrow & N_1 & \xrightarrow{g_1} & N_2 & \xrightarrow{g_2} & N_3 & \longrightarrow & 0. \end{array}$$

   Prove that there is an exact sequence

$$0 \to \mathrm{Ker}(h_1) \to \mathrm{Ker}(h_2) \to \mathrm{Ker}(h_3).$$