Algebra Preliminary Examination
August 2016

Instructions:
- Write your student ID number at the top of each page of your exam solution.
- Write only on the front page of your solution sheets.
- Start each question on a new sheet of paper. Each question is worth 10 points.
- In answering any part of a question, you may assume the results of previous parts.
- To receive full credit, answers must be justified.
- In this exam "ring" means "ring with identity $1 \neq 0$" and "module" means "unital module".
- This exam has two pages.

1. Let R be a Principal Ideal Domain (P.I.D.) and let S be a ring with identity. Let $f : R \to S$ be a surjective ring homomorphism.

(a) Prove that every ideal in S is principal.

(b) Give an example which demonstrates that S need not be a P.I.D.

2. Let R be a commutative ring and let I and J be ideals of R.

(a) Prove that every element of $R/I \otimes_R R/J$ can be written as a simple tensor of the form $(1 \mod I) \otimes_R (r \mod J)$.

(b) Prove that there is an R-module isomorphism $R/I \otimes_R R/J \cong R/(I + J)$ mapping $(r \mod I) \otimes_R (r' \mod J)$ to $rr' \mod (I + J)$.

3. Let R be a commutative ring and S be a multiplicatively closed subset of R. Prove that the prime ideals of $S^{-1}R$ are in one-to-one correspondence ($\wp \leftrightarrow S^{-1}\wp$) with the prime ideals of R which do not intersect with S.

4. Determine if the matrices

$$A = \begin{bmatrix} 2 & -2 & 14 \\ 0 & 3 & -7 \\ 0 & 0 & 2 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} 2 & 2 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & 3 \end{bmatrix}$$

have the same rational canonical form over $\mathbb{Q}$.

5. Let R be an integral domain. Recall that an R-module N is called *torsion free* if $\mathrm{Tor}(N) = 0$ where

$$\mathrm{Tor}(N) := \{x \in N \mid rx = 0 \text{ for some non-zero } r \in R\}.$$

Prove that if M is a projective R-module then M is torsion free.

6. Let V and W be vector spaces over the field F and let $T : V \to W$ be a linear transformation. Assume that V is finite-dimensional. Prove that

$$\dim_F V = \dim_F \ker(T) + \dim_F T(V),$$

where $\ker(T)$ denotes the kernel of T.

7. Let G be the additive abelian group $\mathbb{Z} \oplus \mathbb{Z}$. Recall that $\langle (a, b) \rangle = \{n(a, b) \in G : n \in \mathbb{Z}\}$ is the cyclic subgroup of G generated by the ordered pair (a, b).

(a) Prove that the quotient group $G/\langle (1, 1) \rangle$ is an infinite cyclic group.

(b) Is the quotient group $G/\langle (2, 2) \rangle$ cyclic? Justify your answer.

8. Classify all groups of order 45.

9. Let K be the splitting field of the polynomial $x^{11} - 3$ over $\mathbb{Q}$. Find the dimension $[K : \mathbb{Q}]$ of the $\mathbb{Q}$-vector space K. Justify your answer.

10. Consider the irreducible polynomial $f(x) = x^5 - 2x^3 - 8x - 2$ shown below. Prove that there exists no radical formula for finding the five complex roots of f.