Algebra Preliminary Examination

August 2009

Directions: Begin each question on a new sheet of paper. All rings are commutative with identity and all modules are unital.

1. Suppose that R is a PID and that P is any prime ideal of R. Prove that the localization R_P is also a PID.

2. Let $\phi : G \longrightarrow H$ be a group epimorphism and let K be a normal subgroup of G.

(a) Prove that $\phi(K)$ is a normal subgroup of H.

(b) Prove that ϕ induces a group homomorphism $\overline{\phi} : G/K \longrightarrow H/\phi(K)$ that is also surjective.

(c) Prove that $\overline{\phi}$ is an isomorphism if and only if $\ker\phi \subset K$.

3. Let p be a prime integer and let ζ_p be a primitive p^{th} root of unity. Prove that the splitting field of the polynomial $x^p - 2 \in \mathbb{Q}[x]$ is $\mathbb{Q}(\sqrt[p]{2}, \zeta_p)$. What is $[\mathbb{Q}(\sqrt[p]{2}, \zeta_p) : \mathbb{Q}]$?

4. Let A, B be finite groups and let p be a prime. Prove that each Sylow p-subgroup of $A \times B$ is of the form $P \times Q$ where $P \in \mathrm{Syl}_p(A)$ and $Q \in \mathrm{Syl}_p(B)$.

5. Let G be a finite group of order n and let $\theta : G \longrightarrow S_n$ be the permutation representation afforded by the action of left-multiplication in G. If $n = rs$ and if $g \in G$ is an element of order r, prove that $\theta(g) = \sigma_1\sigma_2 \cdots \sigma_s$ where each σ_i is an r-cycle.

6. Let G be a nonzero finite abelian group viewed as a $\mathbb{Z}$-module via the usual scalar multiplication. Prove that G does not have the structure of a $\mathbb{Q}$-module.

7. Prove that the polynomial $x^2y^2 + x^2y + y^2 + x + 1$ is irreducible in $\mathbb{Q}[x, y]$.

8. Write down all possible Galois groups for a cubic polynomial in $\mathbb{Q}[x]$. Find the Galois groups of $x^3 + 2x - 4$ and $x^3 - 3x + 1$ over $\mathbb{Q}$. (Hint: If $f(x) = x^3 + px + q$ then $\Delta = -4p^3 - 27q^2$ and $\mathrm{disc}(f) = \Delta^2$.)

9. Prove that $2 \otimes [1] = 0$ in $\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$. Prove that $2 \otimes [1] \neq 0$ in $2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$.

10. Let $F(A)$ be the free R-module on the set A and let $F(B)$ be the free R-module on the set B. In addition, let $|X|$ denote the cardinality of any set X.

(a) Prove that if $|A| = |B|$, then $F(A) \simeq F(B)$.

(b) Prove that if $|A|$ and $|B|$ are finite, then $F(A) \simeq F(B)$ if and only if $|A| = |B|$.