Algebra Prelim January 2013

In this exam, the term "ring" is short for "commutative ring with identity" and "module" means "unital module". The order of a group G is denoted $|G|$. Let R be a ring.

Full credit will only be given for solutions that are completely justified.

1. Let $f\colon A \to B$ be a homomorphism between finite abelian groups. Assume that $|A|$ and $|B|$ are relatively prime, and prove that $f = 0$.

2. Let G be a group with operation written multiplicatively. Recall that a *composition series of length* n for G is a chain of subgroups $\{1\} = G_0 \subset G_1 \subset \cdots \subset G_{n-1} \subset G_n = G$ such that for $i = 1, \ldots, n$ the subgroup G_{i-1} is normal in G_i such that the quotient G_i/G_{i-1} is simple. Let p be a prime number, and assume that G is a p-group such that $|G| = p^m$. Prove that G has a composition series of length m.

3. Let C, D, and E be $n \times n$ matrices over $\mathbb{C}$ such that E is invertible and $C = EDE^{-1}$. Prove that $\mathbb{C}^n$ has a basis consisting of eigenvectors for C if and only if $\mathbb{C}^n$ has a basis consisting of eigenvectors for D.

4. Let $K \subseteq L$ be a finite field extension.
   (a) Prove that if $[L : K] = 2$, then the extension $K \subseteq L$ is normal.
   (b) Prove or give a counterexample: if $[L : K]$ is prime, then the extension $K \subseteq L$ is normal.

5. Give an example of a non-separable finite field extension.

6. Prove that every euclidean domain is a principal ideal domain.

7. Prove or give a counterexample: If R is a principal ideal domain, then so is the polynomial ring $R[X]$.

8. Let M be an R-module and fix a submodule $N \subseteq M$. Define $(N : M) := \{r \in R \mid rM \subseteq N\}$. Prove that $(N : M)$ is an ideal of R.

9. Let $0 \to L \to M \to N \to 0$ be an exact sequence of R-modules and R-module homomorphisms, and let $P \subset R$ be a prime ideal of R.
   (a) Prove that $M = 0$ if and only if $L = 0 = N$.
   (b) Prove that the localization $M_P = 0$ if and only if $L_P = 0 = N_P$.

10. Let M be an R-module, and set $\operatorname{Ann}_R(M) = (0 : M) = \{r \in R \mid rM = 0\}$. (See Question 8.) For each $r \in R$, let $\mu_r\colon M \to M$ be given by multiplication by r, that is, $\mu_r(m) := rm$.
    (a) Prove that μ_r is an R-module homomorphism for each $r \in R$.
    (b) Prove that the map $\chi\colon R \to \operatorname{Hom}_R(M, M)$ given by $\chi(r) = \mu_r$ is an R-module homomorphism.
    (c) Prove that $\operatorname{Ker}(\chi) = \operatorname{Ann}_R(M)$.