Algebra Preliminary Examination

June 2009

Directions: Begin each question on a new sheet of paper. All rings are commutative with identity and all modules are unital.

1. Prove that the ring $\mathbb{Z}[x]$ is not a PID. Is $\mathbb{Z}[x]$ a UFD? Briefly justify your answer.

2. Suppose that $p(x) \in F[x]$ is irreducible with $\deg(p) = n$ and suppose that K/F is a finite field extension with $[K : F] = m$. Prove that if $\gcd(m, n) = 1$, then $p(x)$ is irreducible over K.

3. Let p, q be distinct prime integers and let G be a group of order p^2q. Prove that G has a normal Sylow subgroup and prove that G is solvable.

4. Let G be a group and let $N \trianglelefteq G$ with the natural epimorphism $\eta : G \longrightarrow G/N$. Prove that for every group homomorphism $f : G \longrightarrow H$ such that $N \leq \operatorname{Ker}(f)$, there exists a unique group homomorphism $\theta : G/N \longrightarrow H$ such that $f = \theta \circ \eta$

5. Let G be an abelian group and let $H \leq G$. A homomorphism $\phi : G \longrightarrow G$ is called idempotent if $\phi \circ \phi = \phi$.

a) Prove that if $\phi : G \longrightarrow G$ is idempotent, then $G = \operatorname{Im}\phi \oplus \operatorname{Ker}\phi$.

b) Prove that H is a direct summand of G if and only if there exists an idempotent homomorphism $\phi : G \longrightarrow G$ such that $H = \operatorname{Im}\phi$.

6. Let R be a ring and let I be an ideal of R. Prove that the following conditions are equivalent:

a) $I = 0$,

b) $I_P = 0$ for each prime ideal $P < R$,

c) $I_M = 0$ for each maximal $M < R$.

7. Let R be a ring and let Σ be the set of all proper ideals of R that consist only of zero-divisors. Prove that Σ has maximal elements with respect to inclusion and that all maximal elements are prime ideals.

8. Prove the following statements:

a) $\mathbb{Q}(\sqrt{2+\sqrt{2}}) = \mathbb{Q}(\sqrt{2-\sqrt{2}})$,

b) The Galois group $\operatorname{Gal}(\mathbb{Q}(\sqrt{2+\sqrt{2}})/\mathbb{Q}) \simeq \mathbb{Z}/4\mathbb{Z}$,

c) There is exactly one subfield that is properly between $\mathbb{Q}$ and $\mathbb{Q}(\sqrt{2+\sqrt{2}})$.

9. Let R be a ring with ideals I, J. Prove that there is an R-module isomorphism $R/I \otimes_R R/J \simeq R/(I + J)$

10. Prove that a non-zero finite abelian group is not projective as a $\mathbb{Z}$-module.