PH.D. PRELIMINARY EXAMINATION IN ALGEBRA

SAMPLE TEST 1995

Notation. In the following examination R denotes a ring. The integers are denoted by $\mathbb{Z}$, and the rational numbers are denoted by $\mathbb{Q}$. We denote by p a prime number of $\mathbb{Z}$.

Problems.

(1) Show that every group of order p^2 is Abelian.

(2) Let p be the smallest prime dividing the order of a finite group. Show that every subgroup of index p is normal.

(3) Every p-group is nilpotent.

(4) Let R be a commutative Noetherian ring. Show that $R[x]$ is a Noetherian ring.

(5) Let $\phi : M \to N$ be a monomorphism of R-modules, let E be an injective R-module and let

$$\overline{\phi} : Hom_R(N, E) \to Hom_R(M, E)$$

be the map given by $\overline{\phi} : g \mapsto f \circ \phi$. Show that $\overline{\phi}$ is an epimorphism of Abelian groups.

(6) Show that every finite division ring is a field.

(7) If $k \subseteq E \subseteq K$ is a tower of fields with e/k Galois and K/E Galois and every $\sigma \in Aut_k(E)$ is extendable to K then show K/k is Galois.

(8) What is the Galois group of $x^5 - 2$ over $\mathbb{Q}$?

(9) Define the tensor product $M \otimes N$ of the R-modules M and N.

(10) If M is a simple R-module, show that there is an epimorphism $\phi : R \to M$.