PH.D. PRELIMINARY EXAMINATION IN ALGEBRA

SUMMER 1994

Notation. In the following examination R denotes a ring. The integers are denoted by $\mathbb{Z}$, and the rational numbers are denoted by $\mathbb{Q}$.

Problems.

(1) Describe the isomorphism classes of the groups of order 8.

(2) Show that no group of order 84 is simple.

(3) Show that if G is a solvable group with subgroup H then both H and G/H are solvable.

(4) Show that $\mathbb{Z}[x]$ is a unique factorization domain but not a principal ideal domain.

(5) Show that the elements of the field $\mathbb{Q}(\sqrt{5})$ that are roots of monic polynomials with integral coefficients are exactly the elements of the ring:

$$\mathbb{Z}\left[\frac{1+\sqrt{5}}{2}\right].$$

(6) Show that if M and P are projective R-modules then $M \otimes_R P$ is a projective R-module.

(7) Show that if R is a Noetherian ring then $R[x]$ is a Noetherian ring.

(8) Show that if every R-module is a direct sum of simple R-modules then every simple R-module is isomorphic to a direct summand of R.

(9) Show that if R is a commutative ring then the free R-modules of rank m is isomorphic to the free R-modules of rank n if and only if $m = n$.

(10) Find the Galois group associated to the polynomial $x^3 - x + 1$ over the field $\mathbb{Q}$.