Algebra Prelim June 2012

In this exam, the term "ring" is short for "commutative ring with identity" and "module" means "unital module".

1. Let I and J be ideals of a ring R such that $I + J = R$. Prove directly (without using the Chinese Remainder Theorem) that there is a ring isomorphism $R/(I \cap J) \xrightarrow{\cong} R/I \times R/J$.

2. Let I and J be ideals of a ring R. Prove that $I \cup J$ is an ideal of R if and only if $I \subseteq J$ or $J \subseteq I$.

3. Let R be a ring, and let $r \in R$. Let $R[X]$ be the polynomial ring in one variable over R, and let R_r be the localization $S^{-1}R$ where $S = \{1, r, r^2, \ldots\}$. Prove that there is a ring isomorphism $R[X]/(rX - 1) \cong R_r$.

4. Let R be an integral domain. Prove that if R is injective as an R-module, then R is a field.

5. Let R be a principal ideal domain (PID), and let M be a finitely generated R-module. Prove that if M is flat over R, then it is free over R.

6. Let K be a field, and let $K(X)$ be the field of fractions of the polynomial ring $K[X]$ in one variable. Prove that $K(X)$ is generated as a $K[X]$-module by the set

$$\{1\} \bigcup \{1/(f(X))^n \mid f(X) \in K[X] \text{ is a monic and irreducible, and } n \in \mathbb{N}\}.$$

   Explain how this is related to partial fraction decompositions.

7. Let V and W be finite dimensional subspaces of a vector space over a field F. Prove that

$$\dim_F(V) + \dim_F(W) = \dim_F(V + W) + \dim_F(V \cap W)$$

   where $\dim_F(-)$ is the vector space dimension over F.

8. Let A be an $n \times m$ matrix with entries in a field k. Let the "row rank" of A be the vector space dimension of the span of the rows of A. Let the "column rank" of A be the vector space dimension of the span of the columns of A. Prove that the row rank of A equals the column rank of A.

9. Prove that there is no simple group of order 36.

10. Prove that the Galois group of the algebraic closure of a finite field must be abelian.