ALGEBRA PRELIMINARY EXAMINATION

MAY 2005

NOTES. $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ are the integers, the rational numbers, the real numbers, and the complex numbers respectively. All rings have identity unless specifically indicated otherwise.

(1) Show that there is no simple group of order 500.

(2) Let G be a finite group and $H \subsetneq G$ a proper subgroup. Show that $\bigcup_{x \in G} x^{-1}Hx \subsetneq G$.

(3) Prove that any group of order p^2 (p a positive prime integer) is abelian.

(4) Let I be an injective $R-$module and $J \subseteq I$ a submodule. Show that J is injective if and only if J is a direct summand of I.

(5) Let R be a commutative ring with identity with the property that for every nonzero ideal the quotient ring R/I is finite. Show that either R is a field or every nonzero prime ideal of R is maximal.

(6) Let R be commutative with identity and $\mathfrak{P} \subseteq R$ an ideal. Show that the ideal $\mathfrak{P}[x] \subseteq R[x]$ is prime if and only if $\mathfrak{P}$ is a prime ideal of R.

(7) Find all possible Jordan canonical forms of a 4×4 matrix over $\mathbb{Q}$ that is annihilated by the polynomial $x^2 - 6x + 9$. For each form that you find, compute its minimal polynomial.

(8) Let F be the splitting field over $\mathbb{Q}$ of the polynomial $x^4 - 2x^2 + 3$. Compute $\mathrm{Gal}(F/\mathbb{Q})$.

(9) Let R be commutative with identity and $J(R)$ the Jacobson radical of R. Show that $x \in J(R)$ if and only if $1 + rx$ is a unit in R for all $r \in R$ (we define the Jacobson radical to be the intersection of all maximal ideals of R).

(10) Let R be an integral domain. Show that the following conditions are equivalent.
  a) Every $R-$module is projective.
  b) Every $R-$module is free.
  c) R is a field.