ALGEBRA PRELIMINARY EXAMINATION

JANUARY 2006

NOTES. $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ are the integers, the rational numbers, the real numbers, and the complex numbers respectively. All rings have identity unless specifically indicated otherwise, and all $R-$modules are unitary.

(1) Let $p \in \mathbb{N}$ be prime. Show that any group of order p^2 is abelian.

(2) Show that any group of order 280 is not simple.

(3) Let a finite group G acts transitively on a finite set Ω of cardinality greater than one. Show that there is an element of G that fixes no element of Ω.

(4) Let R be a commutative ring with 1 and I an injective $R-$module. Show that if the sequence

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is exact, then the sequence

$$0 \longrightarrow \operatorname{Hom}_R(C, I) \xrightarrow{f^*} \operatorname{Hom}_R(B, I) \xrightarrow{g^*} \operatorname{Hom}_R(A, I) \longrightarrow 0$$

is also exact.

(5) Let R be a commutative ring with 1 and let J be the intersection of all maximal ideals of R.
   (a) Show that if $x \in J$ and $r \in R$ then $1 + rx$ is a unit in R.
   (b) Show that if M is a finitely generated $R-$module with $M = JM$ then $M = 0$.

(6) Let M be a simple left $R-$module. Show that a homomorphism $f: M \longrightarrow M$ is either an isomorphism or the zero homomorphism and hence $\operatorname{End}_R(M)$ is a division ring.

(7) Suppose I is a proper ideal of a domain R that is injective as a $R-$module, show $I = 0$.

(8) Show that if R is an integral domain with the property that R/I is a finite ring for any nonzero ideal I, then every nonzero prime ideal of R is maximal.

(9) Find the minimal polynomial over $\mathbb{Q}$ of the element $\sqrt{2 + \sqrt{2}} \in \bar{\mathbb{Q}}$ and find the Galois group of the Galois closure of $\mathbb{Q}[\sqrt{2 + \sqrt{2}}]$ over $\mathbb{Q}$.

(10) Show for a field K of characteristic $p > 0$ that the following are equivalent:
   (a) Every finite field extension of K is separable.
   (b) The Frobenius homomorphism $F: K \longrightarrow K$ given by $F: x \mapsto x^p$ is an epimorphism.