ALGEBRA PRELIMINARY EXAMINATION

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Abstract. In this examination all fields are commutative. All rings contain 1 and all modules are unitary. The rational numbers are denoted by $\mathbb{Q}$. If I and J are ideals of a commutative ring then $(I : J)$ is the ideal of elements x of the ring such that $xJ \subseteq I$. The spectrum of a commutative ring is the topological space with the Zariski topology whose elements are the prime ideals of the ring.

(1) Let M, N be R-modules and $f, g \in \mathrm{Hom}_R(M, N)$. Show that there exists a module E and a homomorphism $e \in \mathrm{Hom}_R(E, M)$ such that $f \circ e = g \circ e$ in $\mathrm{Hom}_R(E, N)$ and such that given any other R-module K with homomorphism $k \in \mathrm{Hom}_R(K, M)$ with the property that $f \circ k = g \circ k$ in $\mathrm{Hom}_R(K, N)$ then there exists a homomorphism $i \in \mathrm{Hom}_R(K, E)$ so that the diagram

$$\begin{tikzcd} K \arrow[r, "k"] \arrow[d, "i"'] & M \arrow[r, "f"] \arrow[d, equal] & N \arrow[d, equal] \\ E \arrow[r, "e"] & M \arrow[r, "g"] & N \end{tikzcd}$$

commutes.

(2) Show that every group of order 105 is solvable.

(3) Show that if p is a prime integer there are no non-Abelian groups of order p^2.

(4) Find the Galois group of $x^3 - x - 1$ over $\mathbb{Q}$.

(5) Let R be a commutative ring, $\mathfrak{p}$ and element of $\mathrm{Spec}(R)$ and M an R-module. Show that

$$R_{\mathfrak{p}} \otimes_R M \cong M_{\mathfrak{p}}.$$

(6) Let k be a field and $f(x) = a_0x^3 + a_1x^2 + a_2x + a_3 \in k[x]$ with $a_0 \neq 0$. Show that f has distinct roots in every extension of k if and only if

$$\begin{vmatrix} a_0 & a_1 & a_2 & a_3 & 0 \\ 0 & a_0 & a_1 & a_2 & a_3 \\ 3a_0 & 2a_1 & a_2 & 0 & 0 \\ 0 & 3a_0 & 2a_1 & a_2 & 0 \\ 0 & 0 & 3a_0 & 2a_1 & a_2 \end{vmatrix} \neq 0.$$

(7) Let k be a perfect field of positive characteristic. Define an action of $k[x]$ on an extension field E that is finite over k by

$$x \cdot v = F(v)$$

where $F \colon E \longrightarrow E$ is the Frobenius homomorphism. Find the decomposition of E into irreducible $k[x]$-modules when $E = \mathbb{F}_{81}$ and $k = \mathbb{F}_3$.

(8) Let R be a commutative domain and M a R-module. Let $\mathrm{Max}(R)$ be the set of maximal ideals of R and $M_{\mathfrak{p}} = R_{\mathfrak{p}} \otimes_R M$ the localization of M at the prime ideal $\mathfrak{p}$. Show that there is a monomorphism

$$M \longrightarrow \prod_{\mathfrak{m} \in \mathrm{Max}(R)} M_{\mathfrak{m}}.$$

(9) Let R be a commutative domain. Show that the only idempotents of R are 0 and 1.

(10) Let R be a commutative Noetherian domain with unique maximal ideal $\mathfrak{m}$. An ideal $\mathfrak{q}$ of R is called irreducible if whenever there are ideals $\mathfrak{q}_i$ with

$$\mathfrak{q} = \mathfrak{q}_1 \cap \mathfrak{q}_2$$

then $\mathfrak{q} = \mathfrak{q}_1$ or $\mathfrak{q} = \mathfrak{q}_2$.

Suppose that $\mathfrak{q}$ is an $\mathfrak{m}$-primary ideal show that if $(\mathfrak{q} : \mathfrak{m})/\mathfrak{q}$ is a one dimensional $R/\mathfrak{m}$-vector space then $\mathfrak{q}$ is irreducible.