ALGEBRA PRELIMINARY EXAMINATION

JANUARY 2007

Abstract. In this examination all fields are commutative. All rings contain 1 and all modules are unitary. The rational numbers are denoted by $\mathbb{Q}$ and the real numbers are denoted by $\mathbb{R}$.

(1) Let G be a group and $\mathrm{Inn}(G)$ the inner automorphisms of G (that is, the automorphisms of G induced by conjugation by an element of G). Show that

$$\mathrm{Inn}(G) \cong G/Z(G)$$

where $Z(G)$ denotes the center of G.

(2) Show that there is no subgroup of A_6 of order 90.

(3) Show that if R is a finite commutative ring (not necessarily with identity), then every element of R is either a unit or a zero divisor.

(4) Show that the commutative ring, R, is Noetherian if and only if every prime ideal of R is finitely generated.

(5) Show that if R is a domain with every nonzero prime ideal invertible, then R is Noetherian.

(6) Show that there is no finite subgroup of the multiplicative group $\mathbb{R}^*$ of order exceeding two.

(7) Let R be a commutative ring with identity with I an injective $R-$module and P a projective $R-$module. Show that the $R-$module $I \otimes_R \mathrm{P}$ is an injective $R-$module.

(8) Let F be the splitting field of the polynomial $x^3 - 7$ (over the rational numbers $\mathbb{Q}$). Find all fields properly contained between $\mathbb{Q}$ and F and prove that your list is complete.

(9) Find the Galois group of the polynomial $x^4 - 2x^2 - 1$ over
  a) the real numbers $\mathbb{R}$ and
  b) the rational numbers $\mathbb{Q}$.

(10) Let F be a field of characteristic $p > 0$ and z an element of a field extension of F that is algebraic over F. Show that z is separable over F if and only if $F(z) = F(z^{p^n})$ for all $n \geq 1$.