ALGEBRA PRELIMINARY EXAMINATION

JUNE 2011

(1) Show that no group of order 72 is simple.
(2) Let G be a finite group of order n and let p be a prime dividing n. Show that the intersection of the Sylow $p-$subgroups is normal in G.
(3) Let G be a simple group of order 168. Show that G is isomorphic to a subgroup of S_8.
(4) Let R be a commutative ring with identity with the property that for any nonzero ideal $I \subseteq R$, the quotient ring R/I is finite. Show that any nonzero prime ideal of R is maximal.
(5) Let R be a commutative ring with identity. Consider the following short exact sequence of $R-$modules:
$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0 .$$
Show that if A and C are finitely generated, then so is B.
(6) Give an example of $R-$modules A, B, M and an injective $R-$module homomorphism $f : A \longrightarrow B$ such that the induced homomorphism $f \otimes 1_M : A \otimes_R M \longrightarrow B \otimes_R M$ is not injective. (You are free to choose the ring R.)
(7) Show any finitely generated $\mathbb{Z}-$submodule of $\mathbb{Q}$ is cyclic.
(8) Find the Galois group of the polynomial $x^4 - 2x^2 - 2$ over $\mathbb{Q}$.
(9) Let F be a field and K an algebraic extension field of F. Suppose that D is an integral domain such that $F \subseteq D \subseteq K$. Show that D is a field.
(10) We say that an integral domain R is ACCP (ascending chain condition on principal ideals) if there is no infinitely ascending chain of principal ideals. Show that if R is ACCP, then every nonzero nonunit of R can be factored into irreducible elements.