Algebra Preliminary Examination

September 2015

Instructions:
- Write your student ID number at the top of each page of your exam solution.
- Write only on the front page of your solution sheets.
- Start each question on a new sheet of paper. Each question is worth 10 points.
- Submit solutions to questions from Part A and from **either Part B or Part C**.
- In answering any part of a question, you may assume the results in previous parts.
- To receive full credit, answers must be justified.
- In this exam "ring" means "ring with identity" and "module" means "unital module".
- This exam has two pages.

A. Rings, Modules, and Linear Algebra (required)

1. Let R be a non-zero commutative ring, M be an R-module, and $S \subseteq R$ be a multiplicatively closed subset. Prove that there exists a unique isomorphism

$$f : S^{-1}R \otimes_R M \to S^{-1}M$$

for which

$$f((r/s) \otimes m) = rm/s$$

for all $r \in R, m \in M$ and $s \in S$.

2. Let R be a Principal Ideal Domain. Prove that every submodule of R^n is free of rank $\leq n$.

3. Determine if the matrices

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$$

have the same rational canonical form over $\mathbb{Q}$.

4. Prove that the rings $\mathbb{Q}[x, y]/(y^2 - x^3)$ and $\mathbb{Q}[t^2, t^3]$ (where x, y and t are indeterminates) are isomorphic.

5. Prove that if R is a Principal Ideal Domain (P.I.D.) and D is a multiplicatively closed subset of R, then $D^{-1}R$ is also a P.I.D.

PARTS B AND C ARE ON PAGE 2.

B. Groups, Fields, and Galois Theory (option 1)

1. Let $\varphi : G \to H$ be a group homomorphism and assume that H is abelian. Prove that if N is a subgroup of G containing $\ker \varphi$, then N is a normal subgroup of G.

2. For a prime p, let $\mathrm{Syl}_p(X)$ denote the set of all Sylow p-subgroups of a group X. Prove the following statements for finite groups G and H whose orders are divisible by p.

(a) If $P \in \mathrm{Syl}_p(G)$ and $Q \in \mathrm{Syl}_p(H)$, then $P \times Q \in \mathrm{Syl}_p(G \times H)$.

(b) If $\mathcal{A} \in \mathrm{Syl}_p(G \times H)$, then $\mathcal{A} = P' \times Q'$ for some $P' \in \mathrm{Syl}_p(G)$ and $Q' \in \mathrm{Syl}_p(H)$.

3. Let F be a field and let $f(x) \in F[x]$ with $\deg(f) = n$. Prove that if K is the splitting field of f over F, then $[K : F]$ divides $n!$.

4. Let $F \subseteq K$ be an extension of fields with $u \in K$ transcendental over F. Prove that if L is a field such that $F \subseteq L \subseteq F(u)$ with $F \neq L$, then u is algebraic over L.

5. Suppose that $f \in \mathbb{Q}[x]$ with $\deg(f) = 5$. Let $K/\mathbb{Q}$ be the splitting field of f and suppose that $\mathrm{Gal}(K/\mathbb{Q}) = A_5$. Does there exist a field L between $\mathbb{Q}$ and K such that $[L : \mathbb{Q}] = 2$? Justify your answer.

C. Homological Algebra (option 2)

1. Let R be a commutative ring, M, N be R-modules. Prove that

$$\mathrm{pd}_R(M \oplus N) = \sup\{\mathrm{pd}_R(M), \mathrm{pd}_R(N)\}.$$

2. Let $0 \to M_1 \to M \to M_2 \to 0$ be an exact sequence of R modules. Prove that

$$\mathrm{pd}_R M_2 \leq 1 + \max\{\mathrm{pd}_R M_1, \mathrm{pd}_R M\}.$$

3. Let M and N be modules over a commutative domain R. Then for all $n \geq 1$, $\mathrm{Tor}_n^R(M, N)$ is torsion.

4. Let R be a noetherian commutative ring and let I be a proper ideal in R. Suppose that I is generated by a regular sequence on R. Prove that I/I^2 is a free (R/I)-module.

5. Let R be a commutative ring and $0 \to A \to B \to C \to 0$ a short exact sequence of R-modules. Assume that $\mathrm{Ext}_R^1(C, A) = 0$. Prove that $B \cong A \oplus C$.