Preliminary Examination (Math 721)
August 2021

Instructions:

- Write your student ID number at the top of each page of your exam solution.
- Write only on the front page of your solution sheets.
- Start each question on a new sheet of paper.
- In answering any part of a question, you may assume the results of previous parts.
- To receive full credit, answers must be justified.

1. Let H be a subgroup of a group G such that $[G : H] = 2$. Prove that H contains every element of G of odd order.

2. Recall that V_4 is the Klein 4-group, Q_8 is the quaternion group, and D_4 is the dihedral group acting on the vertices of a square. **Prove or disprove** each statement.

   (a) $D_4 \simeq V_4 \rtimes_\theta \mathbb{Z}_2$ for some group homomorphism $\theta : \mathbb{Z}_2 \to \mathrm{Aut}(V_4)$.

   (b) $Q_8 \simeq V_4 \rtimes_\theta \mathbb{Z}_2$ for some group homomorphism $\theta : \mathbb{Z}_2 \to \mathrm{Aut}(V_4)$.

3. Let H be a subgroup of a group G such that $[G : H]$ is finite. Prove that G has a normal subgroup N such that $N \subseteq H$ and $[G : N]$ is finite. **Hint:** Consider the left regular action $G \times \mathcal{L}_H \to \mathcal{L}_H$.

4. Classify all groups of order $539 = 7^2 \cdot 11$.

5. Let F be a field and let $f(x) \in F[x]$ be an irreducible polynomial. Prove that if $g(x) \in F[x]$ is any polynomial and $p(x)$ is any irreducible factor of the composition $f(g(x))$, then $\deg(f)$ divides $\deg(p)$.

6. Let K be the splitting field for the polynomial $f(x) = x^8 - 2$ over the field $\mathbb{Q}$ of rational numbers.

   (a) Determine the degree $[K : \mathbb{Q}]$ of the field extension $\mathbb{Q} \subseteq K$.

   (b) Now let $F = \mathbb{Q}(\sqrt[4]{2})$. Determine the Galois group $\mathrm{Gal}(K/F)$.

7. Let $p < q$ be primes and let K be the splitting field of some irreducible polynomial $f(x) \in \mathbb{Q}[x]$. Prove that if $[K : \mathbb{Q}] = pq$, then there exists a normal field extension $\mathbb{Q} \subseteq E$ such that $[E : \mathbb{Q}] = p$.