Algebra Preliminary Examination
January 2019

INSTRUCTIONS:
• Write your student ID number at the top of each page of your exam solution.
• Write only on the front page of your solution sheets.
• Start each question on a new sheet of paper. Each question is worth 10 points.
• For this exam, you have **two options**:

- Option 1: Submit solutions to questions from Part A and from Part B.
- Option 2: Submit solutions to questions from Part A and from Part C.

• In answering any part of a question, you may assume the results in previous parts.
• To receive full credit, answers must be justified.
• In this exam "ring" means "ring with identity" and "module" means "unital (unitary) module". If $\phi : R \to S$ is a ring homomorphism, we also assume $\phi(1_R) = 1_S$.
• This exam has two pages.

A. Rings, Modules, and Linear Algebra (required)

1. Let G be $\mathbb{Z}$-module generated by v_1, v_2, v_3 subject to the relations

$$\begin{aligned} -4v_1 + 2v_2 - 2v_3 &= 0 \\ -6v_1 + 10v_2 + 4v_3 &= 0 \\ -12v_1 + 6v_2 - 6v_3 &= 0 \end{aligned}$$

Find the invariant factors of G.

2. Let R be a commutative ring and let I, J be ideals of R such that R/IJ has no nonzero nilpotent elements. Prove that $IJ = I \cap J$.

3. Let R be a commutative ring, M an R-module, and $f : M \to M$ an R-module homomorphism such that $f \circ f = f$. Prove that $M = \operatorname{Ker} f \oplus \operatorname{Im} f$ (internal direct sum).

4. Let $A \subseteq B$ be an extension of integral domains such that the induced extension of quotient fields $Q(A) \subseteq Q(B)$ is finite. Let $S = A \setminus \{0\}$. Prove that $S^{-1}B = Q(B)$.

5. Let $R = \mathbb{Z}[i\sqrt{5}]$.

(a) Show that 3 is an irreducible element of R.

(b) Prove that the elements 6 and $2 + 2i\sqrt{5}$ do not have a greatest common divisor.

6. Let A be an integral domain with fraction field K. For an A-module T, we denote as usual $\operatorname{Tor}(T) = \{x \in T \mid ax = 0 \text{ for some } a \in A \setminus \{0\}\}$. Let M be a finitely generated A-module and let r denote the maximal number of linearly independent elements in M.

(a) Prove that $r = \dim_K(K \otimes_A M)$.

(b) Prove that there exists a free A-submodule N of M such that $\operatorname{rank} N = r$ and $\operatorname{Tor}(M/N) = M/N$.

(c) Assume that $\operatorname{Tor}(M) = (0)$. Prove that there exists an injective A-module homomorphism $M \to A^r$.

B. Groups, Fields, and Galois Theory (option 1)

1. Recall that $D_8 = \langle r, s : r^4 = e = s^2, rs = sr^{-1} \rangle$ is the dihedral group of order 8 acting on the verices of a square, and that $Q_8 = \{\pm\mathbf{1}, \pm\mathbf{i}, \pm\mathbf{j}, \pm\mathbf{k}\}$ is the Hamiltonian quaternion group of order 8. **Prove or disprove**:

(a) The symmetric group S_4 contains a subgroup isomorphic to D_8.

(b) The symmetric group S_4 contains a subgroup isomorphic to Q_8.

2. Let p and q be (not necessarily distinct) primes. Prove that a finite group of order p^2q must be solvable.

3. Let $u \in \mathbb{C}$ be the complex number $u = \sqrt{1 - \sqrt[3]{2}}$. Determine the degree $[\mathbb{Q}(u) : \mathbb{Q}]$ of the field extension $\mathbb{Q} \subseteq Q(u)$ and find the (irreducible) minimal polynomial $m_{u,\mathbb{Q}}(x) \in \mathbb{Q}[x]$.

4. Let K be the splitting field of the polynomial $p(x) = x^3 - 7$ over the field $\mathbb{Q}$ of rational numbers. Exhibit with proof all intermediate fields between $\mathbb{Q}$ and K.

C. Homological Algebra (option 2)

1. Let R be a commutative ring and I an ideal of R.

(a) Assume that R/I is a flat R-module. Prove that $I = I^2$.

(b) Assume that R/I is a projective R-module. Prove that there exists $e \in R$ idempotent such that $I = (e)$.

2. Let $0 \to M_1 \to M \to M_2 \to 0$ be an exact sequence of R modules. Assume that $\operatorname{pd}_R M < \sup\{\operatorname{pd}_R M_1, \operatorname{pd}_R M_2\}$. Prove that

$$\operatorname{pd}_R M_2 = \operatorname{pd}_R M_1 + 1.$$

3. Let R be a commutative ring, M an R-module, and a_1, a_2 an M-regular sequence.

(a) Prove that a_1 is a non-zero-divisor on M/a_2M.

Assume, in addition, that $I := (a_1, a_2) \subseteq \operatorname{Jac}(R)$ and M is finitely generated.

(b) Prove that a_2, a_1 is an M-regular sequence.

4. Let R be a commutative ring and let $0 \to A \to B \to C \to 0$ be an exact sequence of finitely generated R-modules. Let I be an ideal of R contained in its Jacobson radical. Prove that

$$\operatorname{depth}_I C \geq \min\{\operatorname{depth}_I A - 1, \operatorname{depth}_I B\}.$$