Algebra Preliminary Examination

22 January 2021

INSTRUCTIONS:

- Write your student ID number at the top of each page of your exam solution.
- Write only on the front page of your solution sheets.
- Start each question on a new sheet of paper. Each question is worth 10 points.
- In answering any part of a question, you may assume the results in previous parts.
- To receive full credit, answers must be justified.
- In this exam "ring" means "ring with identity" and "module" means "unital (unitary) module". If $\phi : R \to S$ is a ring homomorphism, we also assume $\phi(1_R) = 1_S$.
- This exam has two pages.

A. Rings, Modules, and Linear Algebra

1. Let $R = \mathbb{Z}[X, Y]$ and the ideal $I = (5, X^2 + 2)$ in R.

(a) Prove that I is a prime ideal of R and that R/I is a PID.

(b) Give an explicit example of a maximal ideal of R that contains I. (Give a set of generators for such an ideal.)

2. Prove that $\dfrac{\mathbb{Q}[X]}{(X)} \otimes_{\mathbb{Q}[X]} \dfrac{\mathbb{Q}[X]}{(X^2+1)} = (0)$.

3. Let R be a commutative ring, I an ideal of R, and M an R-module. Let

$$(0 :_M I) = \{x \in M \mid ix = 0 \text{ for all } i \in I\}.$$

Prove that $(0 :_M I)$ is a submodule of M and we have an isomorphism of R-modules

$$\operatorname{Hom}_R(R/I, M) \cong (0 :_M I).$$

4. Let R be a commutative ring and let N be a submodule of an R-module M. Assume that M is finitely generated and M/N is a projective R-module. Prove that N is finitely generated.

5. Let G be a $\mathbb{Z}$-module with generators x, y, z subject to the relations $x + 2y + 5z = 0$ and $3x + 3y + 9z = 0$. Find the the elementary divisors of the $\mathbb{Z}$-module G.

6. Let $R = \mathbb{Z}[i\sqrt{5}]$.

(a) Show that 3 is an irreducible element of R.

(b) Prove that the elements 6 and $2 + 2i\sqrt{5}$ do not have a greatest common divisor.

B. Homological Algebra

1. Let K be a field and $R = K[x]/(x^2)$. For each integer $i \geq 0$, compute

(a) $\mathrm{Ext}^i_R(R, R/(x))$

(b) $\mathrm{Ext}^i_R(R/(x), R)$

(c) $\mathrm{Ext}^i_R(R/(x), R/(x))$.

2. Let $R = k[x, y]$ where k is a field, and let $I = (x, y)R$.

(a) Show that

$$0 \to R \xrightarrow{\phi} R \oplus R \xrightarrow{\psi} R \to k \to 0$$

where $\phi(a) = (-ya, xa), \psi(a, b) = xa + yb$ for $a, b \in R$ is a projective resolution of the R-module $k \cong R/I$.

(b) Compute $\mathrm{Tor}^R_i(I, k)$ for all $i \geq 0$.

(c) Show I that is not a flat R-module.

3. Let R be a commutative ring and let $0 \to K \to P \to A \to 0$ be a short exact sequence of R-modules with P projective. Assume that A is not projective. Prove that

$$\mathrm{pd}_R A = \mathrm{pd}_R K + 1.$$

4. Let R be a commutative ring and let

$$0 \to A \to B \to C \to 0$$

be a short exact sequence of R-modules. Assume that $\underline{x} = x_1, \ldots, x_n$ is a sequence that is both A-regular and C-regular. Prove that $\underline{x}$ is B-regular, too.