**Algebra Preliminary Examination**
January 2018

Instructions:

- Write your student ID number at the top of each page of your exam solution.
- Write only on the front page of your solution sheets.
- Start each question on a new sheet of paper.
- In answering any part of a question, you may assume the results of previous parts.
- To receive full credit, answers must be justified.
- In this exam "ring" means "commutative ring with identity" and "module" means "unital module". If $\varphi : R \to S$ is a ring homomorphism, then $\varphi(1) = 1$.
- This exam has two pages.

1. Are the quotient rings $\mathbb{Z}[x, y]/(x^2 - y)$ and $\mathbb{Z}[x, y]/(x^2 - y^2)$ isomorphic? Justify your answer.

2. Let $R$ be the quadratic integer ring $\mathbb{Z}[\sqrt{-5}]$.

   (a) Prove that $\gcd(6, 2 + 2\sqrt{-5})$ is nonexistent in $R$.

   (b) Is $R$ a Euclidean domain? Justify your answer.

3. Is the polynomial $f(x, y) = xy^2 + x^2y + 2xy + y + x + 1$ irreducible in $\mathbb{Z}[x, y]$? Justify your answer.

4. Let $G$ be a free $\mathbb{Z}$-module of finite rank and let $H$ be a submodule of $G$. Prove that the quotient module $G/H$ is finite if and only if $\text{rank}(H) = \text{rank}(G)$.

5. Let $R$ be a ring.

   (a) Suppose that $R = I \oplus J$ is the internal direct sum of ideals $I, J \subseteq R$. Prove that $I$ is principally generated by some idempotent element $e \in I$.

   (b) Assume now that $R$ is an integral domain. Prove that every $R$-module is projective if and only if $R$ is a field.

6. Prove or disprove the following statements. Feel free to use the usual calculus for tensor products where appropriate.

   (a) $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}$ and $\mathbb{Q} \otimes_{\mathbb{Q}} \mathbb{Q}$ are isomorphic as $\mathbb{Q}$-vector spaces.

   (b) $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ and $\mathbb{C} \otimes_{\mathbb{C}} \mathbb{C}$ are isomorphic as $\mathbb{R}$-vector spaces.

7. Recall that $J_k(\lambda)$ is the $k \times k$ Jordan block with eigenvalue $\lambda$. Let $V = \mathbb{Q}^8$ and let $T : V \to V$ be the linear transformation represented by the the $8 \times 8$ matrix $A$ given by.

$$A = \begin{pmatrix} J_3(1) & & & 0 \\ & J_2(1) & & \\ & & J_2(-1) & \\ 0 & & & J_1(-1) \end{pmatrix}.$$

   (a) Is the matrix $A$ diagonalizable?

   (b) Recall that $V^T$ is the the $\mathbb{Q}$-vector space $V$ endowed with the $\mathbb{Q}[x]$-module action given by $x \cdot \mathbf{v} = T(\mathbf{v})$. Find the invariant factor direct sum decomposition of the $\mathbb{Q}[x]$-module $V^T$.

   (c) Find the minimal polynomial $m_A(x) \in \mathbb{Q}[x]$ of the matrix $A$.

   (d) Find the characteristic polynomial $\chi_A(x) \in \mathbb{Q}[x]$ of the matrix $A$.

   (e) Find the rational canonical form for the matrix $A$.

8. Recall that if $V$ is a vector space, then $\mathrm{Aut}(V)$ is the set of all invertible linear transformations $V \to V$. Let $p$ be a prime and let $V$ be a finite dimensional vector space over $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$. Prove that if $G$ is a $p$-subgroup of $\mathrm{Aut}(V)$, then there exists a nonzero vector $\mathbf{v} \in V$ such that $T(\mathbf{v}) = \mathbf{v}$ for all linear transformations $T \in G$. **Hint:** Use the Fixed Point Theorem for $p$-group actions.

9. Classify all groups of order $1225 = 5^2 \cdot 7^2$.

10. Let $\omega \in \mathbb{C}$ be a primitive $5^{\text{th}}$ root of unity.

   (a) Compute the Galois group of $\mathbb{Q}(\omega)$ over $\mathbb{Q}$.

   (b) Exhibit (with proof) the complete lattice of subfields of $\mathbb{Q}(\omega)$.