# Algebra Preliminary Examination

August 2007

- Begin each question on a new sheet of paper.
- In answering any part of a question, you may assume the results in previous parts, even if you have not solved them.

**All rings have identity and all modules are unitary.**

**1.** Let $G$ be a group and $H$ a subgroup of $G$ (not necessarily normal subgroup) with $[G : H] = n$. Prove that for every $g \in G$ we have $g^{n!} \in H$.

**2.** For a group $G$, we denote $Z(G) = \{x \in G \mid xy = yx \text{ for all } y \in G\}$.

(a) Prove that $Z(G)$ is a normal subgroup of $G$.

(b) Let $G$ be a group with $G/Z(G)$ cyclic. Prove that $G$ is abelian.

(c) Let $G$ be a non-abelian group with $p^3$ elements, where $p > 2$ is a prime number. Prove that $x \to x^p$ defines a group homomorphism from $G$ to $Z(G)$.

**3.** Let $G$ be a finite group. For an element $g \in G$, define the centralizer $C(g)$ of g to be

$$C(g) = \{h \in G \mid gh = hg\}.$$

(a) If $g$ and $g'$ are conjugate to each other (i.e. $g = hg'h^{-1}$ for some $h \in G$), prove that $C(g)$ and $C(g')$ are subgroups of $G$ with the same number of elements.

(b) Let $g_1, g_2, \ldots, g_l$ be a complete set of representatives from the conjugacy classes of $G$ ($l$ is called the class number of $G$). Prove that

$$\frac{1}{|C(g_1)|} + \frac{1}{|C(g_2)|} + \cdots + \frac{1}{|C(g_l)|} = 1.$$

(c) Find all the finite groups with the class number $l = 3$.

**4.** Let $R$ be a commutative ring and $I$ an ideal of $R$. Show that if $R/I$ is a projective $R$-module, then $I$ is a principal ideal generated by an idempotent element (i.e. an element $e$ such that $e^2 = e$).

**5.** Let $R$ be commutative ring and $J(R)$ the Jacobson radical of $R$. Show that $x \in J(R)$ if and only if $1 + rx$ is a unit in $R$ for all $r \in R$. (We define the Jacobson radical to be the intersection of all maximal ideals of $R$).

**6.** Let $F$ be a field and $E = F(c)$ a finite separable field extension of $F$. Let $K \supset E$ be a splitting field of the minimal polynomial of $c$ over $F$. Prove that for every prime $p$ dividing the degree $[K : F]$ there exists a field $L$ between $F$ and $K$ such that $[K : L] = p$ and $K = L(c)$.

**7.** (a) Prove that the ring $R = \mathbb{Z}[\sqrt{-2}]$ is Euclidean.

(b) Show that $R/(3 + 2\sqrt{-2})$ is a field. What is the characteristic of this field?

**8.** Find the Galois group of the extension $\mathbb{Q} \subset K$, where $K$ is the splitting field over $\mathbb{Q}$ of $X^4 - 3X^2 + 4$.

**9.** Let $n > 2$ be an integer. Prove that in $\mathbb{Z}[\sqrt{-n}]$, 2 is irreducible but not a prime. Is the same statement true for $n \in \{1, 2\}$?

**10.** Let $K$ be an algebraically closed field. Prove that $K$ has infinitely many elements.