Algebra Preliminary Examination
August 2018

INSTRUCTIONS:
- Write your student ID number at the top of each page of your exam solution.
- Write only on the front page of your solution sheets.
- Start each question on a new sheet of paper. Each question is worth 10 points.
- For this exam, you have **two options**:
  - Option 1: Submit solutions to questions from Part A and from Part B.
  - Option 2: Submit solutions to questions from Part A and from Part C.
- In answering any part of a question, you may assume the results in previous parts.
- To receive full credit, answers must be justified.
- In this exam "ring" means "ring with identity" and "module" means "unital (unitary) module". If $\phi : R \to S$ is a ring homomorphism, we also assume $\phi(1_R) = 1_S$.
- This exam has two pages.

A. Rings, Modules, and Linear Algebra (required)

1. Let $\mathbb{Z}$ be the ring of integers and let X be an indeterminate over $\mathbb{Z}$. How many elements does the ring $\mathbb{Z}[X]/(X^2 - 3, 2X + 4)$ have? Justify your answer.

2. Let R be a commutative ring and I, J ideals of R. If R/I and R/J are noetherian rings, prove that $R/(I \cap J)$ is a noetherian ring.

3. Let R be a commutative ring and M an R-module such that

(a) $M \neq (0)$ and

(b) (0) and M are the only R-submodules of M.

Prove that there exists a maximal ideal $\mathfrak{m}$ of R such that M is isomorphic (as an R-module) to $R/\mathfrak{m}$.

4. (a) Prove that the ring $R = \mathbb{Z}[\sqrt{-2}]$ is Euclidean.

(b) Show that $R/(3 + 2\sqrt{-2})$ is a field. What is the characteristic of this field?

5. Let R be commutative ring and $J(R)$ the Jacobson radical of R. Show that $x \in J(R)$ if and only if $1 + rx$ is a unit in R for all $r \in R$. (Recall that the Jacobson radical is the intersection of all maximal ideals of R).

6. Let A be a $\mathbb{Z}$-module with generators x, y, z subject to the relations $x + 2y + 5z = 0$ and $3x + 3y + 9z = 0$. Find the elementary divisors of the $\mathbb{Z}$-module A.

B. Groups, Fields, and Galois Theory (option 1)

1. Let H be a finite subgroup of a group G and let $\mathcal{L}_H = \{aH : a \in G\}$ be the set of left cosets of H in G. Consider the left regular action $H \times \mathcal{L}_H \to \mathcal{L}_H$ given by $h \cdot aH = haH$. Recall that the normalizer of H in G is the subgroup $N_G(H) = \{g \in G : gHg^{-1} = H\}$.

(a) Prove that $\text{Fix}(\mathcal{L}_H) = N_G(H)/H$ and conclude that $|\text{Fix}(\mathcal{L}_H)| = [N_G(H) : H]$.

(b) Prove that if G is a finite p-group and H a proper subgroup, then H is properly contained in $N_G(H)$.

2. Prove that a group of order $300 = 2^2 \cdot 3 \cdot 5^2$ is not simple.

3. Let $f(x) \in F[x]$ with splitting field K over F. Define the *multiplicity* of the root $u \in K$ to be the integer n such that $f(x) = (x-u)^n g(x)$ where $(x-u) \nmid g(x)$ in $K[x]$. Prove that if f is irreducible over F, then any two of its roots $u, v \in K$ have the same multiplicity.

4. Let K be the splitting field of the polynomial $x^4 - 2x^2 - 2 \in \mathbb{Q}[x]$.

(a) Determine the cardinality of the galois group $\text{Gal}(K/\mathbb{Q})$.

(b) Find an automorphism $\sigma \in \text{Gal}(K/\mathbb{Q})$ of order 4.

C. Homological Algebra (option 2)

1. Let R be a commutative local noetherian ring and P a finitely generated projective R-module. Prove that P is a free R-module.

2. Let I, J be ideals in a commutative ring R such that $I + J = R$. Prove that

$$\text{Tor}_1^R(R/I, R/J) = 0.$$

3. Let R be a commutative ring and $0 \to A \to B \to C \to 0$ a short exact sequence of R-modules. Assume that $\text{Ext}_R^1(C, A) = 0$. Prove that $B \cong A \oplus C$ (isomorphism of R-modules).

4. Let R be a commutative ring, M, N be R-modules, and let $I = \text{Ann}_R N = \{x \in R \mid xN = 0\}$.

(a) Assume that I contains a regular element on M. Prove that $\text{Hom}_R(N, M) = 0$.

(b) Assume that x is a regular element on M with $x \in I$. Prove that

$$\text{Hom}_R(N, M/xM) \cong \text{Ext}_R^1(N, M).$$