Algebra Preliminary Examination
January 2008

• BEGIN EACH QUESTION ON A NEW SHEET OF PAPER.
• IN ANSWERING ANY PART OF A QUESTION, YOU MAY ASSUME THE RESULTS IN PREVIOUS PARTS, EVEN IF YOU HAVE NOT SOLVED THEM.

All rings have identity and all modules are unitary.

1. Let R be a commutative ring and I, J, K ideals of R. Assume that $K \subseteq I \cup J$. Prove that either $K \subseteq I$ or $K \subseteq J$.

2. Let R be a commutative ring and let I be an ideal of the polynomial ring $R[X]$. Suppose that for some monic polynomial $f \in I$, $\deg(f) \leq \deg(g)$ for all nonzero $g \in I$. Prove that I is a principal ideal.

3. (a) Let $(\mathbb{Q}, +)$ be the abelian additive group of rational numbers. Prove that every finitely generated subgroup of $(\mathbb{Q}, +)$ is cyclic.

(b) Prove that $(\mathbb{Q}, +)$ is not a finitely generated group.

4. Let a be an element of a commutative ring R such that a is idempotent, i.e., $a^2 = a$.

(a) Prove that the principal ideal aR is a ring with identity element a.

(b) Let $b = 1 - a$. Prove that b is an idempotent and establish a ring isomorphism

$$R \cong (aR) \times (bR).$$

5. Let $\mathbb{Z}$ be the ring of integers and let X be an indeterminate over $\mathbb{Z}$. How many elements does the ring $\mathbb{Z}[X]/(X^2 - 3, 2X + 4)$ have? Justify your answer.

6. Let $\mathbb{Z}$ be the ring of integers and let X be an indeterminate over $\mathbb{Z}$. Is every ideal of $\mathbb{Z}[X]/(X^2 - 1)$ principal? Justify your answer.

7. Let G be a finite group of order n and let d be an integer relatively prime to n.

(a) Prove that there exists an integer k such that every $x \in G$ satisfies $x^{kd} = x$.

(b) Show that for every $y \in G$ there exists a unique $x \in G$ such that $x^d = y$.

8. (a) Let p, q be two prime numbers such that $p < q$ and p does not divide $(q - 1)$. Let G be a finite group with pq elements. Prove that G is cyclic.

(b) Let G be a group with $595 = 5 \times 7 \times 17$ elements. Prove that G has a **normal** subgroup with 17 elements.

9. Let R be an integral domain of characteristic $p > 0$. Let $F : R \to R$ be the function given by $F(a) = a^p$ and for $n > 0$ denote $F^n = F \circ F \circ \ldots \circ F$ (n times). Prove that :

(a) p is a prime number;

(b) F is a ring homomorphism;

(c) For $n > 0$, the set $\{a \in R \mid F^n(a) = a\}$ is a finite **field**;

(d) If R is finite, then R is a field;

(e) If R is finite, then F is bijective;

(f) If R is finite, then there exists a positive integer n such that $F^n : R \to R$ is the identity.

10. Let $\mathbb{Q}$ be the field of rational numbers and let $\mathbb{Q} \subseteq F$ be a field extension. Let $\sigma : F \to F$ be a nonzero ring homomorphism.

(a) Prove that if F is algebraic over $\mathbb{Q}$, then σ is an isomorphism.

(b) Show that the conclusion might fail if F is not algebraic over $\mathbb{Q}$.