Algebra Preliminary Examination

June 2015

Instructions:
- Write your student ID number at the top of each page of your exam solution.
- Write only on the front page of your solution sheets.
- Start each question on a new sheet of paper. Each question is worth 10 points.
- Submit solutions to questions from Part A and from **either Part B or Part C**.
- In answering any part of a question, you may assume the results in previous parts.
- To receive full credit, answers must be justified.
- In this exam "ring" means "ring with identity" and "module" means "unital module".
- This exam has two pages.

A. Rings, Modules, and Linear Algebra (required)

1. Prove or disprove: If R is a commutative ring, and I, J are ideals of R, then the set $\{ij \mid i \in I \text{ and } j \in J\}$ is an ideal of R.

2. Prove or disprove: every UFD is a PID.

3. Prove or disprove: If R is a commutative ring, and M is a finitely generated R-module, then every submodule $N \subseteq M$ is finitely generated over R.

4. Prove or disprove: the matrices $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 9 & 2 & 0 & 0 \\ 8 & 7 & 3 & 0 \\ 6 & 5 & 4 & 4 \end{pmatrix}$ and $\begin{pmatrix} 4 & 0 & 0 & 0 \\ 7 & 3 & 0 & 0 \\ 8 & 6 & 2 & 0 \\ 4 & 11 & 5 & 1 \end{pmatrix}$ have the same rational canonical forms over $\mathbb{Q}$.

5. Assume that R is a PID, and let M, N be finitely generated R-modules. Let $\mathrm{Hom}_R(M, N)$ denote the set of R-module homomorphisms $M \to N$. Prove that $\mathrm{Hom}_R(M, N)$ is finitely generated over R.

PARTS B AND C ARE ON PAGE 2.

B. Groups, Fields, and Galois Theory (option 1)

1. Find a representative of each conjugacy class in S_6.

2. Let $f\colon G \to A$ be a homomorphism of groups such that A is abelian. Let $[G,G]$ be the commutator subgroup of G, and let $p\colon G \to G/[G,G]$ be the natural group epimorphism. Prove that there is a unique group homomorphism $f'\colon G/[G,G] \to A$ such that $f = f' \circ p$.

3. Let $f(x) = x^4 - 12x^2 + 18 \in \mathbb{Q}[x]$.

(a) Find a splitting field K of $f(x)$ over $\mathbb{Q}$ and determine $[K : \mathbb{Q}]$

(b) Prove that $\mathbb{Q} \subseteq K$ is a Galois extension with Galois group $\mathrm{Gal}(K : \mathbb{Q})$ isomorphic to $\mathbb{Z}/4\mathbb{Z}$.

4. Prove that if the Galois group of a splitting field of a cubic polynomial over $\mathbb{Q}$ is the cyclic group of order 3 then all of the roots of the cubic polynomial are real.

5. Let α and β be two complex numbers with minimal polynomials $f(x)$ and $g(x)$, respectively, over $\mathbb{Q}$. Prove that $f(x)$ is irreducible over $\mathbb{Q}(\beta)$ if and only if $g(x)$ is irreducible over $\mathbb{Q}(\alpha)$.

C. Homological Algebra (option 2)

1. Let R be a commutative ring and M an R-module. Assume that we have the exact sequences of R-modules $P \xrightarrow{\pi} M \to 0$ and $P' \xrightarrow{\pi'} M \to 0$ where P and P' are projective R-modules. Prove that

$$\ker \pi \oplus P' \cong \ker \pi' \oplus P.$$

2. Let I, J be ideals in a commutative ring R. Prove that

$$\mathrm{Tor}_1^R(R/I, R/J) \cong \frac{I \cap J}{IJ}.$$

3. Let R be a principal ideal domain and let $a, b \in R\setminus\{0\}$. Show that $\mathrm{Ext}_R^1(R/aR, R/bR) = 0$ if and only if a and b are relatively prime.

4. Let R be a commutative ring, M, N be R-modules, and let $I = \mathrm{Ann}_R N = \{x \in R \mid xN = 0\}$.

(a) Assume that I contains a regular element on M. Prove that $\mathrm{Hom}_R(N, M) = 0$.

(b) Assume that $x_1, x_2, \ldots, x_n$ is a regular sequence on M with $x_1, x_2, \ldots, x_n \in I$. Prove that

$$\mathrm{Hom}_R(N, M/(x_1, \ldots, x_n)M) \cong \mathrm{Ext}_R^n(N, M).$$

5. Let R be a commutative ring and let $0 \to K \to P \to A \to 0$ be a short exact sequence of R-modules with P projective. Assume that A is not projective. Prove that

$$\mathrm{pd}_R A = \mathrm{pd}_R K + 1.$$