Algebra Preliminary Examination
June 2014

• In answering any part of a question, you may assume the results in previous parts

All rings have identity and all modules are unitary (unital).

1. Let G be a group with a normal subgroup N such that $G/N \simeq \mathbb{Z}$. Prove that for each positive integer m, there exists a normal subgroup $N(m) \trianglelefteq G$ such that $G/N(m) \simeq \mathbb{Z}/m\mathbb{Z}$.

2. Classify all groups of order 99.

3. Let K be the splitting field of the polynomial $x^4 + 1 \in \mathbb{Q}[x]$ and let G be the Galois group of K over $\mathbb{Q}$.

   (a) Prove that G is isomorphic to the Klein 4-group $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.

   (b) Determine all subfields of K and prove that your determination is complete.

4. Let K/F be a field extension with $\alpha \in K$. Prove that if $[F(\alpha) : F]$ is finite and odd, then $F(\alpha) = F(\alpha^2)$.

5. (a) Prove that 7 is a prime element of $\mathbb{Z}[i]$.

   (b) Prove that $(7, X^2 + 1)$ is a maximal ideal of $\mathbb{Z}[X]$.

6. Let R be a commutative ring with a unique maximal ideal. Prove that 0 and 1 are the only idempotents in R. (An element $x \in R$ is said to be idempotent if $x^2 = x$.)

7. Let $R = k[X, Y]$ be a polynomial ring over the field k and $I = (X, Y)$ the ideal of R generated by the indeterminates X and Y.

   (a) Let $H : I \times I \to R$ be the function defined by $H(a, b) = \frac{\partial a}{\partial X}(0, 0) \cdot \frac{\partial b}{\partial Y}(0, 0)$ where $\frac{\partial a}{\partial X}$ and $\frac{\partial b}{\partial Y}$ are formal partial derivatives. Prove that H is an R-bilinear map.

   (b) Prove that $X \otimes Y \neq Y \otimes X$ in $I \otimes_R I$.

   (c) Show that there exists a unique R-module homomorphism $\theta : I \otimes_R I \to R \otimes_R R$ such that $\theta(a \otimes b) = a \otimes b$ for all $a, b \in I$.

   (d) Is the map θ defined in part (c) injective? Justify your answer.

8. Prove that $\mathbb{Q}$ is not a projective $\mathbb{Z}$-module.

9. Let V be a finite dimensional k-vector space (k is a field) and let $\phi : V \to V$ be a k-linear transformation. Prove that there exists a positive integer n such that
$$\mathrm{Im}(\phi^n) \cap \mathrm{Ker}(\phi^n) = (0).$$

10. Let G be an abelian group with three generators v_1, v_2, v_3 subject to the relations
$$6v_1 + 4v_2 + 2v_3 = 0$$
$$-2v_1 + 2v_2 + 6v_3 = 0$$
Prove that $G \cong (\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/10\mathbb{Z}) \oplus \mathbb{Z}$ and find new generators w_1, w_2 and w_3 for G such that $2w_1 = 0$, $10w_2 = 0$ and w_3 has infinite order.