Algebra Preliminary Examination
September 2014

• In answering any part of a question, you may assume the results in previous parts

All rings have identity and all modules are unitary (unital).

1. Write down all nonisomorphic abelian groups of order 72.

2. Let G be a group of order p^2q^2 with p, q distinct primes such that $p \nmid (q^2 - 1)$ and $q \nmid (p^2 - 1)$. Prove that if P is a Sylow p-subgroup and Q is a Sylow q-subgroup, then $G \cong P \times Q$.

3. Let $F \subseteq L \subseteq K$ be a tower of fields with $u \in K$ algebraic over F and let $m_{u,F}(x) \in F[x]$ be the minimum polynomial of u over F. Prove that if $\deg(m_{u,F})$ and $[L : F]$ are relatively prime, then $m_{u,F}(x)$ is irreducible over L.

4. Let K be the splitting field of the polynomial $x^4 - 2x^2 - 2 \in \mathbb{Q}[X]$. Find an automorphism $\sigma \in \mathrm{Gal}(K/\mathbb{Q})$ of order 4.

5. Let K be a field and V a vector space over K. Let α and β be K-endomorphisms of V with $\alpha\beta = 0$ and $\mathrm{id}_V = \alpha + \beta$. Show that $V = \mathrm{im}(\alpha) \oplus \mathrm{im}(\beta)$.

6. Prove that the ideal $(X^2 + 2, X^2 + 7)$ is a maximal ideal of $\mathbb{Z}[X]$.

7. Prove or disprove:

(a) $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}$ is isomorphic to $\mathbb{Q}$ as a $\mathbb{Z}$-module.

(b) $\mathbb{Q}(\sqrt{10}) \otimes_{\mathbb{Q}} \mathbb{Q}(\sqrt{10})$ is isomorphic to $\mathbb{Q}(\sqrt{10})$ as $\mathbb{Q}$-vector space.

8. Construct, up to similarity, all the linear transformations $T : \mathbb{C}^6 \to \mathbb{C}^6$ with minimal polynomial $(X - 5)^2(X - 6)^2$.

9. Let $R = \mathbb{Z}[X]$ and $I = (2, X)$ be the ideal of R generated by 2 and X. Prove that I is not a free R-module and the rank of the R-module I is 1.

10. Let $R = \{f \in \mathbb{Z}[X] \mid \text{the coefficient of } X \text{ in } f \text{ is even}\}$.

(a) Prove that R is a subring of $\mathbb{Z}[X]$.

(b) Prove that 2 and $2X$ have a g.c.d (greatest common divisor) in R but not a l.c.m. (least common multiple) in R.