Analysis Preliminary Examination - June 2008

Throughout these problems, m denotes Lebesgue measure, m^* denotes Lebesgue outer measure, and $\mathcal{M}$ is the collection of Lebesgue measurable sets.

1. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function for which the derivative $f'(x)$ exists everywhere. (f' is not necessarily continuous.) Is it true that

$$f(x) - f(-1) = \int_{-1}^{x} f'(t)dt$$

   for all $x \in (-1, 1)$? Justify! (Hint: Consider $x^2 \cos(1/x^2)$ on $[-1, 1]$.)

2. Let ν be counting measure on $\mathbb{N}$. Let $G : [0, 1] \times \mathbb{N} \to \mathbb{R}$ be given by $G(x, n) = (x/2)^n$.

   (a) Prove that for $0 < a \leq 1$,

$$G^{-1}((-\infty, a)) = \bigcup_{n \in \mathbb{N}} ([0, 2a^{1/n}) \times \{n\}).$$

   (b) Deduce from (a) that G is $dm \times d\nu$-measurable.

   (c) Use Tonelli's theorem to prove that

$$\sum_{n=1}^{\infty} \frac{1}{(n+1)2^n} = 2 \ln 2 - 1.$$

   (Hint: $\frac{1}{n+1} = \int_0^1 x^n dx$.)

3. Let $(X, \mathcal{F}, \mu)$ be a measure space, and let $f : X \to [0, \infty)$ be measurable and non-negative.

   (a) Let $E_m = \{x \in X : f(x) > 1/m\}$ for $m \in \mathbb{N}$. Prove that

$$\lim_{m \to \infty} \int_{E_m} f d\mu = \int_X f d\mu.$$

   (b) Prove that, if $\int_X f\, d\mu < \infty$, then for all $\varepsilon > 0$ there exists $A \in \mathcal{F}$ with $\mu(A) < \infty$ such that

$$\int_X f d\mu < \int_A f\, d\mu + \varepsilon.$$

4. (a) Show that if $A \subset \mathbb{R}$ then there exists $B \in \mathcal{M}$ with $A \subset B$ and

$$m^*(A) = m(B).$$

   (b) Suppose that $A \subset \mathbb{R}$, $m^*(A) < \infty$, and there exists $C \in \mathcal{M}$ such that $C \subset A$ and $m(C) = m^*(A)$. Use the definition of measurability to show that $A \in \mathcal{M}$.

5. Let $(X, \mathcal{F}, \mu)$ be a measure space. Suppose $f_n \in L^1(\mu)$ for all $n \in \mathbb{N}$, f is measurable, and $f_n \to f$ uniformly.

   (a) If $\mu(X) < \infty$ show that $f \in L^1(\mu)$ and $f_n \to f$ in $L^1(\mu)$.

   (b) Show that the statement of (a) may be false if $\mu(X) = \infty$. (Construct a counterexample using Lebesgue measure.)

6. Let dF be the Lebesgue-Stieltjes measure with distribution function

$$F(x) = \begin{cases} 0 & \text{if } x < 0, \\ \frac{x^2}{2} & \text{if } 0 \leq x < 1, \\ 1 & \text{if } 1 \leq x. \end{cases}$$

   (a) Prove that F is not absolutely continuous with respect to Lebesgue measure.

   (b) Find the Radon-Nikodym decomposition of dF with respect to Lebesgue measure, i.e., find $f \in L^1(\mathbb{R}, m)$ and a measure λ with $\lambda \perp m$ so that $dF = f dm + d\lambda$.