Analysis Qualifying Exam **June 2014**

- Submit 6 problems from part 1 and 3 problems from part 2.
- Please start every problem on a new page, label your pages and write your student ID, but not your name, on each page.

Part 1 - Real Analysis

Lebesgue measure is denoted by m.

1. Let $\mathcal{A}$ be an algebra of sets that is closed under countable increasing unions. Show that $\mathcal{A}$ is a σ-algebra.
2. Let $A \subset E \subset B$, where A, B are Lebesgue measurable sets of finite measure. Prove that if $m(A) = m(B)$, then E is measurable.
3. If $\{f_n\}_{n\in\mathbb{N}}$ is a sequence of Lebesgue measurable real-valued functions, prove that $f = \liminf f_n$ is Lebesgue measurable.
4. Let $f : \mathbb{R} \to [0, \infty)$ be Lebesgue measurable.
   (a) Let $E_m = \{x \in \mathbb{R} : f(x) > 1/m\}$. Use the monotone convergence theorem to show that
   $$\lim_{m\to\infty} \int_{E_m} f\,dm = \int_{\mathbb{R}} f\,dm.$$
   (b) Prove that, if $\int_{\mathbb{R}} f\,dm < \infty$, then for all $\varepsilon > 0$ there exists $A \in \mathcal{B}_{\mathbb{R}}$ with $m(A) < \infty$ so that
   $$\int_{\mathbb{R}} f\,dm < \int_A f\,dm + \varepsilon.$$
5. Let $\{f_n\}$ be a sequence of Lebesgue integrable functions that converge to f in L^1.
   (a) Prove that $\{f_n\}$ converges to f in measure.
   (b) Give an example of a sequence $\{f_n\}$ and a function f such that $\{f_n\}$ converges to f in measure, but $\{f_n\}$ does not converge to f in L^1.
6. Let f, g be Lebesgue integrable functions on $\mathbb{R}$. Prove that the function $F(x, y) = f(y)g(x - y)$ is Lebesgue integrable in $\mathbb{R}^2$
7. Let μ_F be the Borel measure on $\mathbb{R}$ with distribution function
   $$F(x) = \begin{cases} \arctan(x) & \text{if } x < 0, \\ x^2 + 1 & \text{if } x \geq 0. \end{cases}$$
   (a) Calculate $\mu_F([0, 3))$ and $\mu_F((0, 3))$.
   (b) State what it means for a measure μ to be absolutely continuous with respect to Lebesgue measure.
   (c) Prove that μ_F is not absolutely continuous with respect to Lebesgue measure.
8. Construct a family of Lebesgue measurable functions $\chi_t : \mathbb{R} \to \mathbb{R}$, $t \in \mathbb{R}$, with the property that $\chi = \sup_{t\in\mathbb{R}} \chi_t$ is not a Lebesgue measurable function. (You may assume without proof that non-measurable sets exist.)

9. Show by way of an example that an open, dense set in $\mathbb{R}$ need not have infinite measure.

10. Let f be a continuous function of bounded variation. Prove that $f = f_1 - f_2$, where both f_1, f_2 are monotonic and continuous.

Part 2 - Functional and Complex Analysis

1. Let $1 \leq p < q < \infty$.

   (a) Give examples that show that neither $L^p(\mathbb{R}) \subset L^q(\mathbb{R})$ nor $L^q(\mathbb{R}) \subset L^p(\mathbb{R})$.

   (b) Prove that if f is bounded, and $f \in L^p(\mathbb{R})$, then $f \in L^q(\mathbb{R})$.

   (c) Prove that if f is supported on a set of finite measure and $f \in L^q(\mathbb{R})$, then $f \in L^p(\mathbb{R})$.

2. Let H be a Hilbert space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

   (a) State Riesz Representation Theorem for Hilbert spaces.

   (b) Let $T : H \to H$ be a bounded operator. Prove that there exists a unique operator $T^* : H \to H$ such that for every $x, y \in H$,

   $$\langle Tx, y \rangle = \langle x, T^* y \rangle.$$

   Hint: For fixed y, define $\Lambda_y : H \to \mathbb{K}$ by $\Lambda_y(x) = \langle Tx, y \rangle$.

3. Let X be a Banach space.

   (a) State the Uniform Boundedness Principle.

   (b) Let $A \subset X$. Prove that A is a bounded set if and only if for every $\Lambda \in X^*$, $\sup\{|\Lambda(a)| : a \in A\} < \infty$.
   Hint: Consider X as a subset of X^{**}.

4. Prove that a finite rank operator is compact.

5. Prove Liouville's theorem, i.e., show that every bounded entire function is constant. (You may use without proof Cauchy's integral formula.)

6. Let f be analytic in the open upper half plane, continuous on the closed upper half plane, and $f(\mathbb{R}) \subseteq \mathbb{R}$. Define $g : \mathbb{C} \to \mathbb{C}$ by

   $$g(z) = \begin{cases} f(z) & \text{if } \Im z \geq 0, \\ \overline{f(\overline{z})} & \text{if } \Im z < 0. \end{cases}$$

   Show that g is entire.(Hint: Show that g is continuous and use Morera's theorem.)

7. Let $f : \mathbb{D} \to \mathbb{D}$, where $D = \{z : |z| < 1\}$. Assume that $f(0) = 0$ and $f'(0) = 1$. Show that $f(z) = cz$ for some constant c with $|c| = 1$. (Hint: Consider $g(z) = z^{-1} f(z)$.)