Analysis Qualifying Exam, September 2015

Submit six of the problems from part 1, and three of the problems from part 2. Start every problem on a new page, label your pages and write your student ID on each page.

Part 1 - Real Analysis
Lebesgue measure is denoted by m.

1. Prove that if $f : X \to \mathbb{R}$ satisfies that the sets $f^{-1}(r, \infty)$ are measurable for every $r \in \mathbb{Q}$, then f is measurable.

2. Let $f \in L^1(0,1)$, and let $h(x,t) = \frac{f(t)}{t}\chi_{\{x \leq t\}}(x,t)$, where $(x,t) \in (0,1) \times (0,1)$. Prove that $h \in L^1((0,1) \times (0,1))$.

3. Let $f_n : [1, \infty) \to \mathbb{R}$ be defined by $f_n(x) = \dfrac{1}{x}\chi_{\{n,\infty)\}}(x)$. Use one of the convergence theorems to study the convergence of this sequence of functions. State the theorem that you are using.

4. (a) State the definition of absolutely continuous measure and give an example.

(b) State Radon-Nikodym's theorem.

5. Let $(X, \mathcal{M}, \mu)$ be a measure space. Prove that there is a σ-algebra $\overline{\mathcal{M}}$ that contains $\mathcal{M}$ and a measure $\overline{\mu}$ on the σ-algebra so that $(X, \overline{\mathcal{M}}, \overline{\mu})$ is complete and $\overline{\mu}|_{\mathcal{M}} = \mu$.

6. Prove using the definition of Lebesgue outer measure that the Lebesgue outer measure is translation invariant (i.e. $m^*(E) = m^*(E + \lambda)$ for any fixed λ in $\mathbb{R}$.

7. If $f \in L^1$ prove that $\{x : f(x) \neq 0\}$ is σ-finite.

8. Suppose that $f_n \to f$ in measure and $g_n \to g$ in measure. Prove that if $\mu(X) < \infty$ then $f_n g_n \to fg$ in measure. Provide an example to show that the condition that $\mu(X) < \infty$ is necessary.

Part 2 - Complex and Functional Analysis
1. Suppose that f is an entire function and for every a the power series

$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n$$

there is c_k equal to zero. Prove that f is a polynomial.

2. Let $P(z) = z^n + a_{n-1}z^{n-1} + \cdots a_1 z + a_0$ be a polynomial over the complex numbers. Use the maximum modulus theorem to prove that $P(z)$ has a zero at some point in $\mathbb{C}$.

3. Let γ be the positively oriented unit circle and compute

$$\frac{1}{2\pi i}\int_\gamma \frac{e^z - e^{-z}}{z^4}\, dz.$$

4. Let $f \in L^p(0,1)$ for $1 \leq p < \infty$. For $F \in L^\infty(0,1)$, we define the multiplication operator, M_F by

$$M_F(f) = F \cdot f,$$

where $\cdot$ denotes the usual multiplication of functions. Show that $M_F\|$ is a bounded operator from $L^p(0,1)$ to $L^p(0,1)$ and compute its operator norm.

5. (a) Let B be a Banach space and $T : B \to B$ a bounded operator. State the definition of the adjoint of T.

(b) Consider the shift operator $S : \ell^2 \to \ell^2$, defined by $S(x_1, x_2, x_3, \ldots) = (0, x_1, x_2, x_3, \ldots)$. Find its adjoint.

(c) Is S a compact operator?

6. (a) State the Uniform Boundedness Principle.

(b) Let X be a Banach space over $\mathbb{R}$, and let $A \subset X$. If for every $f \in X^*$ the set $f(A) = \{f(x) : x \in A\}$ is bounded, show that A is a bounded subset of $\mathbb{R}$.