Analysis Preliminary Examination
August 2017

Submit six of the problems from part 1, and three of the problems from part 2. Start every problem on a new page, label your pages and write your student ID on each page.

Part 1: Real Analysis
Lebesgue measure is denoted m and unless stated otherwise $(X, \mathcal{M}, \mu)$ is a generic measure space.

1. Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space.
   (a) Show that if $E \in \mathcal{M}$ satisfies $\mu(E) > 0$, then there exists $F \subseteq E$, $F \in \mathcal{M}$ such that $0 < \mu(F) < \infty$.
   (b) Prove that if f, g are non-negative measurable functions such that $\int_E f d\mu = \int_E g d\mu$ for every $E \in \mathcal{M}$, then $f = g$ μ-almost everywhere.

2. Let $(X, \mathcal{M}, \mu)$ be a measure space and let $f_n : X \to \mathbb{R}$ be a sequence of measurable functions.
   (a) Assume that there exists a non-negative measurable function F satisfying
   $$\int_X F d\mu < \infty \quad \text{and} \quad |f_n| \leq F \quad \text{for all } n.$$
   Prove that
   $$\int_X \limsup f_n d\mu \geq \limsup \int_X f_n d\mu.$$
   (b) Give an example showing that the conclusion of part (a) may fail without the assumption of the existence of F.

3. Suppose that $A \subseteq \mathbb{R}$ satisfies $m_1(A) = 0$, where m_1 denotes one dimensional Lebesgue measure. Let $f : \mathbb{R} \to \mathbb{R}^2$, satisfying that $|f(x) - f(y)| \leq \sqrt{|x - y|}$ for every $x, y \in \mathbb{R}$. Show that $m_2(f(A)) = 0$ (here, m_2 is the Lebesgue measure in $\mathbb{R}^2$).

4. Let ν be a signed measure defined on $(X, \mathcal{M})$.
   (a) Write the definition of positive, negative and null sets for ν.
   (b) Show that a countable union of positive sets for ν is a positive set for ν.

5. Consider the Lebesgue-Stieltjes measure dF associated to the function
   $$F(x) = \begin{cases} 0 & \text{if } x < 0 \\ \pi/4 & \text{if } 0 \leq x < 1 \\ \arctan x & \text{if } 1 \leq x. \end{cases}$$
   Show that dF is neither absolutely continuous nor mutually singular with respect to the Lebesgue measure m on $\mathbb{R}$, and find the Radon-Nikodym decomposition of dF with respect to m.

6. Let $(X, \mathcal{M}, \mu)$ be a measure space, and let $f, f_n : X \to \mathbb{R}$ $(n \in \mathbb{N})$ be measurable functions.

   (a) Give the definition of convergence in measure.

   (b) Show that $f_n \to f$ in L^1 implies that $f_n \to f$ in measure.

   (c) Give an example to show that the converse of (b) does not always hold.

7. Let $f : X \to \mathbb{R}$ be measurable. Assume there exists measurable sets E_n with $E_n \subseteq E_{n+1}$ and $X = \cup_n E_n$, and $c > 0$ so that $\int_{E_n} |f| \leq c$ for all $n \in \mathbb{N}$. Show that f is in L^1 and $\int |f| \leq c$.

8. Let $f(x) = 1$ for $-1 \leq x \leq 1$ and $f(x) = 0$ otherwise. Define $g(x, y) = f(x - y)$. Is g an element of $L^1(\mathbb{R} \times \mathbb{R})$? Justify.

Part 2: Complex and Functional Analysis

9. Let M be a closed subspace of the Banach space X. Let $x_0 \in X$ be such that the distance from x_0 to M is positive (the distance is defined by $d(x_0, M) = \inf\{\|x_0 - y\| : y \in M\}$). Prove that there exists a functional $F \in X^*$ with the following properties:

   (a) $F(x) = 0$ for every $x \in M$.

   (b) $F(x_0) = d(x_0, M)$.

   (c) $\|F\| = 1$

10. Prove that a necessary and sufficient condition for a normed space X to be complete is that for every sequence of vectors $\{x_n\} \subset X$ such that $\sum_{n=1}^{\infty} \|x_n\| < \infty$, the series $\sum_{n=1}^{\infty} x_n$ converges to an element $x \in X$.

11. Let X be a Banach space and let $T : X \to Y$ be bijective.

    (a) State the Open Mapping Theorem.

    (b) Use the Open Mapping Theorem to prove that if T is continuous then so is T^{-1}.

12. Assume that $f : \mathbb{C} \to \mathbb{C}$ is continuous on $\mathbb{C}$ and analytic on $\mathbb{C} \backslash [-1, 1]$. Show that f is entire.

13. Does there exist a holomorphic surjection from the unit disk to $\mathbb{C}$? (Construct an example, or prove that it cannot exist.)

14. Evaluate

$$\int_\gamma \frac{e^{iz}}{z^2} dz$$

    where γ is the positively oriented circle of radius 1 with center at the origin.