Analysis Preliminary Examination
May 2017

Submit six of the problems from part 1, and three of the problems from part 2. Start every problem on a new page, label your pages and write your student ID on each page.

Part 1: Real Analysis

Lebesgue measure is denoted m and unless stated otherwise $(X, \mathcal{M}, \mu)$ is a generic measure space.

1. Let μ be a measure defined on $(\mathbb{R}, \mathcal{P}(\mathbb{R}))$ such that for every $x \in \mathbb{R}$, $\mu(\{x\}) > 0$. Show that if $\mu(A) < \infty$, then for every $n \in \mathbb{N}$, the set $\{x \in A : \mu(\{x\}) > \frac{1}{n}\}$ is finite. Conclude that if $\mu(A) < \infty$, then A must be countable.

2. Let X be an uncountable set. For each $S \subseteq X$, we define

$$\mu^*(S) = \begin{cases} 0 & \text{if } S \text{ is countable} \\ 1 & \text{if } S \text{ is uncountable} \end{cases}$$

$$\nu^*(S) = \begin{cases} 0 & \text{if } S \text{ is countable} \\ \infty & \text{if } S \text{ is uncountable} \end{cases}$$

   Show that μ^* and ν^* are outer measures and describe the σ-algebra of measurable sets for each of them.

3. Let $(X, \mathcal{M}, \mu)$ be a measure space and let $f : X \to \mathbb{R}$ be a measurable function. Show that

$$\lim_{n\to\infty} \int_X \left[1 - \left(\frac{2}{e^{f(x)} + e^{-f(x)}}\right)^n\right] d\mu(x) = \mu(\{x \in X : f(x) \neq 0\}),$$

   explaining which convergence theorem you are using.

4. True or false? Let A, B be subsets of $\mathbb{R}$ such that B is Lebesgue measurable in $\mathbb{R}$ and $A \times B$ is Lebesgue measurable in $\mathbb{R}^2$. Then A is Lebesgue measurable in $\mathbb{R}$. (If true, prove it; if false, give a counterexample).

5. Let $\{a_n\}$ be a sequence of real numbers such that $\sum_{n=0}^{\infty} |a_n| = M < \infty$. Let $X = \{n \in \mathbb{Z} : n \geq 0\}$ and let ν be the counting measure on $\mathcal{P}(X)$. On the product space $(X \times X, \mathcal{P}(X) \times \mathcal{P}(X), \nu \times \nu)$ we consider the function $f : X \times X \to \mathbb{R}$ defined by $f(m, n) = a_n 2^{-|m-n|}$.

   (a) Prove that f is in $L^1(X \times X)$.

   (b) Show that the integral of f equals $\sum_{n=0}^{\infty} a_n(3 - 2^{-n})$, explaining which theorem(s) you have used.

6. Let $\nu(E) = \int_E f d\mu$, where μ is a positive measure and f is an extended μ-integrable function. Describe the Hahn decomposition of ν and its variations (positive, negative and total) in terms of f and μ.

7. Let $(X, \mathcal{M}, \mu)$ be a measure space. Let ν be another measure on $(X, \mathcal{M})$.

   (a) Prove that if $\mu \perp \nu$ and $\nu \ll \mu$, then $\nu(E) = 0$ for every $E \in \mathcal{M}$.

   (b) Let $\{\lambda_n\}$ be a sequence of measures on $(X, \mathcal{M})$ such that $\lambda_n \ll \mu$ for every n. We define $\lambda(A) = \sum_{n=0}^{\infty} \lambda_n(A)$. Prove that λ is a measure on $(X, \mathcal{M})$ and that $\lambda \ll \mu$.

8. Let $F : \mathbb{R} \to \mathbb{R}$. Prove that the following statements are equivalent.

   (a) There exists a constant $M < \infty$ such that for every $x, y \in \mathbb{R}$, $|F(x) - F(y)| \leq M|x - y|$.

   (b) F is absolutely continuous and $|F'(x)| \leq M$ for almost every $x \in \mathbb{R}$.

Part 2: Complex and Functional Analysis

9. Let f be analytic in an open set G and g be analytic in an open set containing $f(G)$. Assume that $g \circ f$ is constant. Show that either f or g must be constant.

10. Let f be analytic in $B(0,1) \backslash \{0\}$ and nonvanishing. Assume that f'/f has a pole of finite order at $z = 0$.

    (a) Show that there exists $m \in \mathbb{N}$ so that

    $$\frac{f'(z)}{f(z)} = \sum_{n=1}^{m} a_n z^{-n} + g(z)$$

    where g is analytic in $B(0,1)$, and a_1 is an integer.

    (b) Show that there exists $k \in \mathbb{Z}$, an analytic function h on $B(0,1)$ with $h(0) \neq 0$, and a polynomial P so that

    $$f(z) = z^k h(z) e^{P(1/z)}.$$

11. Compute for $a > 0$

    $$\int_0^{\infty} \frac{\log x}{x^2 + a^2} dx.$$

    (Hint: Consider a branch of the logarithm that allows for integration on a contour involving $[-K, K]$.)

12. Let f be analytic on $B(0,1)$ and suppose $|f(z)| \leq 1$ for $|z| < 1$. Show that $|f'(0)| \leq 1$.

13. Consider the spaces $(l_p, \| \ \|_p),\ 1 \leq p \leq \infty$.

   a) Show that $l_p \subset l_q$ properly if $p < q$.

   b) Show that l_p is separable if $1 \leq p < \infty$. What can you say about the separability of l_∞? Justify your answer.

14. Consider $(C[0,1], \| \ \|_\infty)$, and let $K(x,y)$ be a function continuous on the unit square $[0,1]^2$. For any $f \in C[0,1]$, define an operator $T : C[0,1] \to C[0,1]$ by

$$Tf(x) = \int_{[0,1]} K(x,y) f(y) dy.$$

   (a) Show that T is a bounded linear operator on $C[0,1]$.

   (b) What is $\|T\|$? Justify your answer.

15. Let M and N be closed subspaces of a Hilbert space H. Prove that $M \oplus N$ is closed provided that $x \perp y$ for all $x \in M,\ y \in N$.

16. Let X be a normed space and $x_0 \in X,\ x_0 \neq 0$. Prove that $\exists\ f \in X^*$ with $\|f\| = 1$ and $f(x_0) = \|x_0\|$. [Hint: Use Hahn-Banach Theorem.]