Analysis Qualifying Exam
August 17, 2018

Submit six of the problems from part 1, and three of the problems from part 2. Start every problem on a new page, label your pages and write your student ID on each page.

The symbol m denotes Lebesgue measure, and $L^1(\mathbb{R}, m)$ denotes the measurable functions on $\mathbb{R}$ that are Lebesgue integrable.

1 Real Analysis

1. Let $E \subseteq \mathbb{R}$ be a set of finite Lebesgue measure, and let $E_n = \{x \in E : |x| > n\}$, for $n = 1, 2, \ldots$. Prove that $\lim_{n\to\infty} |E_n| = 0$.

2. Let f_j be a real-valued Borel measurable function for $j \in \mathbb{N}$. Show that $g = \sup f_j$ is Borel measurable.

3. Let $f \in L^1(\mathbb{R})$. Show that if
$$\int_a^b f(x)dx = 0$$
for every $a, b \in \mathbb{Q}$, $a < b$, then $f(x) = 0$ for almost every $x \in \mathbb{R}$.

4. Let
$$F(x) = \frac{1}{1+x^2},$$
and let μ be the (signed!) measure with distribution function F.
   (a) Find a Hahn decomposition of $\mathbb{R}$ for μ. (Prove your answer.)
   (b) Find the distribution function of the total variation measure of μ.

5. Let $f \in L^1(\mathbb{R}, m)$. Define $g_n(x) = f(x - \frac{1}{n})$ and prove that g_n converges to f in $L^1(\mathbb{R}, m)$ as $n \to \infty$. (Hint: Prove this first for continuous f.)

6. Let (f_n) be a sequence of functions in $L^1[0, 1]$. For each of the following statements, give a proof if the result is true, or a counterexample if it is false.
   (a) If $\lim_{n\to\infty} \|f_n\|_{L^1} = 0$, then f_n converges to 0 almost everywhere.
   (b) If $\|f_n\|_{L^1} < 2^{-n}$ for each n, then f_n converges to 0 almost everywhere.

7. Let f, g be measurable functions on $\mathbb{R}^n$ such that $fg \in L^1(\mathbb{R}^n)$, and $g \geq 0$. Prove that
$$\int_{\mathbb{R}^n} f(x)g(x)dx = \int_0^\infty \int_{\{x\in\mathbb{R}^n : g(x)>t\}} f(x)dxdt.$$

8. (a) State the Lebesgue differentiation theorem.
   (b) Let E be a Borel set in $\mathbb{R}$. Show that
$$\lim_{r\to 0} \frac{m(E \cap (x-r, x+r))}{m((x-r, x+r))} = \chi_E(x) \text{ a.e.}$$

2 Complex and Functional Analysis

9. (a) Show that there are no non-real solutions of $\cos z = 0$.

   (b) Show that there exist constants a_k so that for all $|z| < \pi/2$

$$\frac{1}{\cos z} = 1 + \sum_{k=1}^{\infty} \frac{a_k}{(2k)!} z^{2k}.$$

10. Set $\mathbb{C}^+ = \{z \in \mathbb{C} : \Im z > 0\}$ and $\mathbb{C}^- = \{z \in \mathbb{C} : \Im z < 0\}$. Assume $f : \overline{\mathbb{C}^+} \to \mathbb{C}$ is analytic in $\mathbb{C}^+$, continuous on the closure $\overline{\mathbb{C}^+}$, and real-valued on $\mathbb{R}$. Define

$$g : \mathbb{C} \to \mathbb{C}$$

by $g(z) = f(z)$ if $\Im z \geq 0$, and $g(z) = \overline{f(\bar{z})}$ if $\Im z < 0$.

   (a) Show that g is analytic in $\mathbb{C}^-$. (Relate power series of g at z_0 with power series of f at $\bar{z}_0$.)

   (b) Show that g is continous everywhere in $\mathbb{C}$.

   (c) Show that g is entire. (Morera!)

11. Show that an analytic function that maps the unit disc into the unit circle is constant.

12. Consider the space c_0 of all real sequences converging to zero.

   (a) State the definition of a separable normed space.

   (b) Prove or disprove: the space c_0 is separable.

13. Let H be a Hilbert space over $\mathbb{C}$ and $T \in B(H)$.

   (a) State the definition of a Hermitian operator $T \in B(H)$.

   (b) Give an example of a Hermitian operator.

   (c) Prove that an operator $T \in B(H)$ is Hermitian if and only if $< Tx, x > \in \mathbb{R}$, $\forall x \in H$.

14. Let X be a normed space over $\mathbb{R}$.

   (a) State the Hahn-Banach Theorem (real version).

   (b) Use the Hahn-Banach Theorem to prove the following statement: If $x_0 \in X$ with $x_0 \neq 0$, then $\exists\ f \in X^*$ with $\|f\| = 1$ and $f(x_0) = \|x_0\|$.