Analysis Qualifying Exam
May 15, 2018

Submit six of the problems from part 1, and three of the problems from part 2. Start every problem on a new page, label your pages and write your student ID on each page.

1 Real Analysis

1. (a) State the definition of Lebesgue outer measure m^*.
   (b) Show that $m^*(E) = m^*(E + \lambda)$ for every $E \subseteq \mathbb{R}$ and $\lambda \in \mathbb{R}$.

2. Let $(X, \mathcal{M}, \mu)$ be a measure space. Prove that there exists a σ-algebra $\overline{\mathcal{M}}$ containing $\mathcal{M}$ and a measure $\overline{\mu}$ on $\overline{\mathcal{M}}$ so that $(X, \overline{\mathcal{M}}, \overline{\mu})$ is complete, and the restriction of $\overline{\mu}$ to $\mathcal{M}$ is μ.

3. Let $f : [0, 1] \to \mathbb{R}$ be a Lebesgue measurable function. Prove that the sets

$$A = \{(x, y) : x \in [0, 1], y \geq f(x)\}$$
$$B = \{(x, f(x)) : x \in [0, 1]\}$$

   are measurable subsets of $[0, 1] \times \mathbb{R}$.

4. Let $f_n(x) = n \sin(x/n)$, and evaluate

$$\lim_{n\to\infty} \int_0^{\pi} f_n(x)dx.$$

5. (a) Give an example of a sequence of Borel measurable functions $\{f_n\}$ on $\mathbb{R}$ that converge uniformly to 0, but such that for every n,

$$\lim_{n\to\infty} \int_{\mathbb{R}} f_n(x)dm(x) = \infty.$$

   (b) Give an example of a sequence of continuous function on $[0, 1]$ such that

$$\lim_{n\to\infty} \int_0^1 f_n(x)dm(x) = 0$$

   but the sequence $\{f_n(x)\}$ is not convergent for any $x \in [0, 1]$.

6. Let $(X, \mathcal{M}, \mu)$ be a measure space, and consider the interval $[0, T]$ with Lebesgue measure. Let $f : X \times [0, T] \to \mathbb{R}$ be a function such that $f(\cdot, t)$ is measurable for every $t \in [0, T]$, and $f(x, \cdot)$ is continuous for each $x \in X$. Assume also that there is a function $g \in L^1(X)$ such that $|f(x, t)| \leq g(x)$ for almost every $x \in X$. Show that the function

$$F(t) = \int_X f(x, t)d\mu(x)$$

   is continuous.

7. Let $f : [a, b] \to \mathbb{R}$ be absolutely continuous. Assume that $E \subset [a, b]$ has measure zero. Prove that $f(E)$ has also measure zero.

8. Let μ_F be the Borel measure on $\mathbb{R}$ with distribution function

$$F(x) = \begin{cases} x & \text{if } x < 0, \\ 1 + e^x & \text{else.} \end{cases}$$

   (a) Calculate $\mu_F([0, 1))$ and $\mu_F((0, 1))$.

   (b) Find the Radon-Nikodym decomposition of μ_F with respect to Lebesgue measure.

2 Complex and Functional Analysis

9. Let f be analytic on $\{z \in \mathbb{C} : |z| < 1\}$ and assume $|f(z)| \leq 1$ for $|z| < 1$. Show that $|f'(0)| \leq 1$.

10. Use contour integration to prove

$$\int_{-\infty}^{\infty} \frac{\log |x|}{1 + x^2} dx = 0.$$

(Hint: Use the branch of the (complex) logarithm that is defined on $\mathbb{C} \setminus \{iy : y \leq 0\}$, and close $[-R, -\varepsilon] \cup [\varepsilon, R]$ in the upper half plane.)

11. Find an analytic function that maps the quarter of the open unit disk given by $\{z \in \mathbb{C} : |z| < 1, \Re z > 0, \Im z > 0\}$ onto the open unit disk in a one-one fashion.

12. A normed space $(X, \| \ \|)$ is called *rotund* if $\forall\ x, y \in X,\ x \neq y$, with $\|x\| = 1 = \|y\|$, then $\|x + y\| < 2$. Prove that every inner product space is rotund.

13. Consider the space $(C[0, 1], \| \ \|_\infty)$ and let $T : C[0, 1] \to C[0, 1]$ be a map defined as, for all $f \in C[0, 1]$,

$$Tf(x) = \int_{[0,1]} K(x, y) f(y) dy,$$

where $K : [0, 1]^2 \to \mathbb{R}$ is a continuous function. Prove that

   (a) T is a bounded linear map.

   (b) What is $\|T\|$? Justify your answer.

14. Let (X, d) be a complete metric space and $D \subset X$ be a dense set. Prove that if $D = \bigcap_n A_n$, where each $A_n \subset X$ is open, then $X \setminus D$ is of first category in X.