Analysis Preliminary Exam
Measure Theory and Integration

Submit six of the following problems. Start every problem on a new page, label your pages, and write your student ID on each page.

1. Let $\mathcal{M}$ be the collection of subsets of $\mathbb{R}$ which are either countable or have countable complement.

   (a) Prove that $\mathcal{M}$ is σ-algebra on $\mathbb{R}$.

   (b) Let $E \in \mathcal{M}$, and define $\mu(E) = 0$ if E is countable and $\mu(E) = 1$ if E has countable complement. Prove that μ is a measure on $\mathcal{M}$.

   (c) Is μ absolutely continuous with respect to the Lebesgue measure m restricted to $\mathcal{M}$? Justify your answer.

2. Consider $\mathbb{R}$ with its usual topological structure.

   (a) Write the definition of Lebesgue outer measure $m^*(A)$ of a set $A \subseteq \mathbb{R}$.

   (b) Prove that if $m^*(A) = 0$ for a set $A \subseteq \mathbb{R}$, then A is Lebesgue measurable.

   (c) Calculate $m^*(\mathbb{Q})$, where $\mathbb{Q}$ is the set of rationals. Justify your work.

3. Let $(X, \mathcal{M}, \mu)$ be a measure space and let $\{f_n\}$ be a sequence of measurable functions.

   (a) Write the definition of convergence in measure of f_n to f.

   (b) Give an example of a sequence f_n and a function f such that f_n converges in measure to f, but it does not converge to f for any $x \in X$.

   (c) Assume that f_n converges to f for almost every $x \in X$, and also assume that $\mu(X) < \infty$. Prove that f_n converges to f in measure.

4. Let $(X, \mathcal{M}, \mu)$ be a measure space.

   (a) Let $\{f_n\}$ be a sequence of measurable functions and assume that there exists a non-negative measurable function F such that $|f_n| \le F$ for every n, and

   $$\int_X F d\mu < \infty.$$

   Prove that

   $$\int_X \limsup f_n d\mu \ge \limsup \int_X f_n d\mu. \tag{1}$$

   (b) Give an example of a sequence f_n for which there is no dominating F and the inequality (1) fails.

5. Compute the following limit, proving everything using appropriate convergence theorems:

$$\lim_{n\to\infty} \int_0^\infty \frac{1 + \frac{x}{\sqrt{\log n}} e^{-\frac{x}{n}}}{(x+1)^2}, dx.$$

6. Let $(X \times Y, \mathcal{M} \times \mathcal{N}, \mu \times \nu)$ be a product measure space.

   (a) State Fubini's theorem.

   (b) Let E and F be measurable subsets of $X \times Y$, such that $\nu(E_x) = \nu(F_x)$ for almost every $x \in X$. Prove that $\mu \times \nu(E) = \mu \times \nu(F)$.

7. Let m be Lebesgue measure in $\mathbb{R}^n$. Prove that for any finite collection of open balls $\{B_1, \ldots, B_N\}$, there exists a disjoint subcollection $\{B_{i_1}, \ldots B_{i_k}\}$ such that

$$m(\cup_{j=1}^{N} B_j) \subseteq 3^n \sum_{\ell=1}^{k} B_{i_\ell}.$$

8. Let f be absolutely continuous on the interval $[\epsilon, 1]$ for every $0 < \epsilon < 1$.

   (a) If f is continuous at 0, does it follow that f is absolutely continuous on $[0, 1]$?

   (b) If f has bounded variation on $[0, 1]$, does it follow that f is absolutely continuous on $[0, 1]$?