Analysis Preliminary Exam
Measure Theory and Integration

Submit six of the following problems. Start every problem on a new page, label your pages, and write your student ID on each page.

1. Let $M$ be the collection of subsets of $\mathbb{R}$ which are either countable or have countable complement.

   (a) Prove that $M$ is $\sigma$-algebra on $\mathbb{R}$.

   (b) Let $E \in M$, and define $\mu(E) = 0$ if $E$ is countable and $\mu(E) = 1$ if $E$ has countable complement. Prove that $\mu$ is a measure on $M$.

   (c) Is $\mu$ absolutely continuous with respect to the Lebesgue measure $m$ restricted to $M$? Justify your answer.

2. Consider $\mathbb{R}$ with its usual topological structure.

   (a) Write the definition of Lebesgue outer measure $m^*(A)$ of a set $A \subseteq \mathbb{R}$.

   (b) Prove that if $m^*(A) = 0$ for a set $A \subseteq \mathbb{R}$, then $A$ is Lebesgue measurable.

   (c) Calculate $m^*(\mathbb{Q})$, where $\mathbb{Q}$ is the set of rationals. Justify your work.

3. Let $(X, \mathcal{M}, \mu)$ be a measure space and let $\{f_n\}$ be a sequence of measurable functions.

   (a) Write the definition of convergence in measure of $f_n$ to $f$.

   (b) Give an example of a sequence $f_n$ and a function $f$ such that $f_n$ converges in measure to $f$, but it does not converge to $f$ for any $x \in X$.

   (c) Assume that $f_n$ converges to $f$ for almost every $x \in X$, and also assume that $\mu(X) < \infty$. Prove that $f_n$ converges to $f$ in measure.

4. Let $(X, \mathcal{M}, \mu)$ be a measure space.

   (a) Let $\{f_n\}$ be a sequence of measurable functions and assume that there exists a non-negative measurable function $F$ such that $|f_n| \leq F$ for every $n$, and

   \[ \int_X F \, d\mu < \infty. \]

   Prove that

   \[ \int_X \limsup f_n \, d\mu \geq \limsup \int_X f_n \, d\mu. \]

   (1)

   (b) Give an example of a sequence $f_n$ for which there is no dominating $F$ and the inequality $1$ fails.

5. Compute the following limit, proving everything using appropriate convergence theorems:

\[ \lim_{n \to \infty} \int_0^\infty \frac{1 + \frac{x}{\sqrt{\log n}} e^{-x/n^2}}{(x+1)^2} \, dx. \]
6. Let \((X \times Y, \mathcal{M} \times \mathcal{N}, \mu \times \nu)\) be a product measure space.

   (a) State Fubini’s theorem.

   (b) Let \(E\) and \(F\) be measurable subsets of \(X \times Y\), such that \(\nu(E_x) = \nu(F_x)\) for almost every \(x \in X\). Prove that \(\mu \times \nu(E) = \mu \times \nu(F)\).

7. Let \(m\) be Lebesgue measure in \(\mathbb{R}^n\). Prove that for any finite collection of open balls \(\{B_1, \ldots, B_N\}\), there exists a disjoint subcollection \(\{B_{i_1}, \ldots, B_{i_k}\}\) such that

\[
m(\bigcup_{j=1}^N B_j) \subseteq 3^n \sum_{\ell=1}^k B_{i_\ell}.
\]

8. Let \(f\) be absolutely continuous on the interval \([\epsilon, 1]\) for every \(0 < \epsilon < 1\).

   (a) If \(f\) is continuous at 0, does it follow that \(f\) is absolutely continuous on \([0, 1]\)?

   (b) If \(f\) has bounded variation on \([0, 1]\), does it follow that \(f\) is absolutely continuous on \([0, 1]\)?