Analysis Preliminary Exam
January 2023

Submit six of the following problems. Start every problem on a new page, label your pages, and write your student ID on each page. **Do not write your name.**

1. Let $E \subset \mathbb{R}$. We define
$$\mu(E) = \sum_{n=1}^{\infty} \chi_E\left(\frac{1}{n}\right)$$
   (a) Prove that μ is a measure defined on $\mathcal{P}(\mathbb{R})$, which is σ-finite but not finite.
   (b) Show that there is no Stieltjes measure dF that is equal to μ.
   (c) Compute the integral $\int_{\mathbb{R}} f\, d\mu$, when $f(x) = e^{-1/x}\chi_{(0,\infty)}$.

2. Let $(X, \mathcal{M}, \mu)$ be a measure space, and let f be a positive integrable function, *i.e.* $\int_X f d\mu < \infty$. Show that for every $\epsilon > 0, \exists A \in \mathcal{M}$, with $\mu(A) < \infty$, such that
$$\int_X f d\mu < \int_A f d\mu + \epsilon.$$

3. For $a \in \mathbb{R}$, $a > 0$, we define the function
$$f(a) = \int_0^{\infty} e^{-at}\frac{\sin t}{t}\, dt.$$
   (a) Prove that $\lim_{a\to\infty} f(a) = 0$. Justify your work using convergence theorems.
   (b) Explain why f is differentiable, specifying the convergence theorems you use and checking that their hypotheses hold for f.
   (c) Find $f'(a)$.

4. Consider the measure space $([0,1]\times[0,1], \mathcal{L}, m_2)$, where $\mathcal{L}$ is the *Lebesgue* σ-algebra and m_2 is the two dimensional Lebesgue measure on $[0,1]\times[0,1]$. Given a set $E \in \mathcal{L}$, we define
$$E_x = \{y \in [0,1] : (x,y) \in E\}, \quad E^y = \{x \in [0,1] : (x,y) \in E\}.$$
   Let m_1 be the Lebesgue measure on $[0,1]$. Prove that if $m_1(E_x) \leq 1/2$ for almost every x, then
$$m_1(\{y \in [0,1] : m_1(E^y) = 1\} \leq 1/2.$$

5. Let $F : \mathbb{R} \to \mathbb{R}$ be the distribution function

$$F(x) \begin{cases} 0, & x < 0 \\ x^2/2, & 0 \leq x < 1 \\ 1, & 1 \leq x. \end{cases}$$

and let dF be the corresponding Lebesgue-Stieltjes measure defined on the Borel σ-algebra of $\mathbb{R}$.

(a) Show that dF is not absolutely continuous with respect to the Lebesgue measure m on $\mathbb{R}$.

(b) Find the Radon-Nikodym decomposition of dF with respect to m, $dF = f dm + d\lambda$, specifying what f and λ are.

6. Let $E \subset \mathbb{R}$ with $m^*(E) > 0$. Show that there exists a bounded subset of E that also has positive outer measure.

7. Let $\{f_n\}$ be a sequence of $\mathbb{R}$-valued measurable functions on $E \in \mathcal{F}$ and $f : E \to \mathbb{R}$ be measurable. Prove that $f_n \to f$ almost everywhere on E if and only if

$$m(\{x \in E : \limsup_n f_n(x) > \liminf_n f_n(x)\}) = 0.$$

8. Show that the function $f(x) = \frac{1}{\sqrt{x}}$ is measurable on $[0,1]$. Calculate $\int_{[0,1]} \frac{1}{\sqrt{x}} dm$. (Notice: f is not Riemann-integrable on $[0,1]$.)

9. (a) State Monotone Convergence Theorem and Fatou's Lemma.

(b) Prove Fatou's Lemma using Monotone Convergence Theorem.

(c) Show by an example that strict inequality in Fatou's Lemma is possible.