Analysis Preliminary Exam
January 2024

Submit six of the following problems. Start every problem on a new page, label your pages, and write your student ID on each page. Do not write your name.

1. Let $g : X \to Y$ be a function. Let $\mathcal{B}$ be a σ-algebra on Y. Show that
$$\mathcal{A} = \{g^{-1}(E) : E \in \mathcal{B}\}$$
is a σ-algebra on X.

2. Let $(X, \mathcal{M})$ be a measurable space and μ be a finitely additive set function on $\mathcal{M}$. Prove that μ is a measure iff it is continuous from below.

3. Let $(X, \mathcal{M})$ be a measurable space and let $X = A \cup B$, where $A, B \in \mathcal{M}$. Prove that a function $f : X \to \mathbb{R}$ is measurable iff f is measurable on A and B.

4. Let $(X, \mathcal{M}, \mu)$ be a measure space.
   (i) State Monotone Convergence Theorem.
   (ii) Assume that If $\{f_n\}$ is a sequence of nonnegative functions on X with $\int f_1 < \infty$. Prove that if $\{f_n\}$ decreases pointwise to a function f, then $\int f = \lim_n f_n(x)$.

5. (a) State the definitions of convergence in L_1 and convergence in measure.
   (b) Prove that if $f_n \xrightarrow{L_1} f$, then $f_n \xrightarrow{m} f$.
   (c) Show by an example that the converse of the statement in (b) is not valid.

6. (a) State Fubini's Theorem.
   (b) Calculate the integral
$$\int_{[0,\frac{\pi}{2}]} \int_{[y,\frac{\pi}{2}]} \frac{y \sin x}{x} dx dy \quad \text{and justify your solution.}$$

7. Consider the function
$$f(x,y) = \begin{cases} \dfrac{xy}{(x^2+y^2)^2} & \text{if } (x,y) \neq (0,0) \\ 0 \text{ if } (x,y) = (0,0). \end{cases}$$
   (a) Show that $\int_{[0,1]}(\int_{[0,1]} f dm(x)) dm(y) = \int_{[0,1]}(\int_{[0,1]} f dm(y)) dm(x)$.
   (b) Prove that f is not m^2-integrable on $[0,1] \times [0,1]$.
   (c) Does the results in (a) and (b) contradict each other? Justify.

8. Let $f : \mathbb{R} \to \mathbb{R}$ be defined as
$$f(x) = \begin{cases} x \sin(\frac{1}{x}) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$
   a) Determine $\overline{D}f(0)$ and $\underline{D}f(0)$.
   b) Prove that f is <u>not</u> of bounded variation on $[0,1]$.
   c) Show that f is uniformly continuous on on $[0,1]$.