Submit six of the following problems. Start every problem on a new page, label your pages, and write your student ID on each page. Do not write your name.

1. Let \( g : X \rightarrow Y \) be a function. Let \( B \) be a \( \sigma \)-algebra on \( Y \). Show that
   \[
   A = \{ g^{-1}(E) : E \in B \}
   \]
   is a \( \sigma \)-algebra on \( X \).

2. Let \( (X, \mathcal{M}) \) be a measurable space and \( \mu \) be a finitely additive set function on \( \mathcal{M} \). Prove that \( \mu \) is a measure iff it is continuous from below.

3. Let \( (X, \mathcal{M}) \) be a measurable space and let \( X = A \cup B \), where \( A, B \in \mathcal{M} \). Prove that a function \( f : X \rightarrow \mathbb{R} \) is measurable iff \( f \) is measurable on \( A \) and \( B \).

4. Let \( (X, \mathcal{M}, \mu) \) be a measure space.
   (i) State Monotone Convergence Theorem.
   (ii) Assume that \( \{ f_n \} \) is a sequence of nonnegative functions on \( X \) with \( \int f_1 < \infty \). Prove that if \( \{ f_n \} \) decreases pointwise to a function \( f \), then \( \int f = \lim_n f_n(x) \).

5. (a) State the definitions of convergence in \( L_1 \) and convergence in measure.
   (b) Prove that if \( f_n \overset{L_1}{\rightarrow} f \), then \( f_n \overset{m}{\rightarrow} f \).
   (c) Show by an example that the converse of the statement in (b) is not valid.

6. (a) State Fubini’s Theorem.
   (b) Calculate the integral
   \[
   \int_{[0, \pi/2]} \int_{[y, \pi/2]} \frac{y \sin x}{x} \, dx \, dy
   \]
   and justify your solution.

7. Consider the function
   \[
   f(x, y) = \begin{cases} 
   \frac{xy}{(x^2 + y^2)^2} & \text{if } (x, y) \neq (0, 0) \\
   0 & \text{if } (x, y) = (0, 0). 
   \end{cases}
   \]
   (a) Show that \( \int_{[0,1]}(\int_{[0,1]} fdm(x))dm(y) = \int_{[0,1]}(\int_{[0,1]} fdm(y))dm(x) \).
   (b) Prove that \( f \) is not \( m^2 \)-integrable on \( [0, 1] \times [0, 1] \).
   (c) Does the results in (a) and (b) contradict each other? Justify.

8. Let \( f : \mathbb{R} \rightarrow \mathbb{R} \) be defined as
   \[
   f(x) = \begin{cases} 
   x \sin(\frac{1}{x}) & \text{if } x \neq 0 \\
   0 & \text{if } x = 0. 
   \end{cases}
   \]
   a) Determine \( Df(0) \) and \( Df(0) \).
   b) Prove that \( f \) is not of bounded variation on \([0, 1]\).
   c) Show that \( f \) is uniformly continuous on \([0, 1]\).