MATH 756 Qualifying Exam

Choose 5 questions. Write all your work and final answers on blank paper. Write your student ID on each paper and include this exam sheet on top.

1. Show that the triangle inequality fails for weak L^p spaces: Find two functions f, g in $L^p[0,1]$ (with Lebesgue measure) such that $[f]_p = 1$, $[g]_p = 1$ and $[f+g]_p > 2$.

2. Consider the 2π-periodic odd function defined on $[0,\pi]$ by $f(x) = x(\pi - x)$.

(a) Draw the graph on f on $[-\pi,\pi]$.

(b) Compute the Fourier coefficients of f and show that

$$f(x) = \frac{8}{\pi} \sum_{k \text{ odd }, k>0} \frac{\sin(kx)}{k^3}.$$

3. Let $F_N(x)$ be the N-th Fejer kernel, *i.e.* $\frac{1}{N+1}\sum_{k=0}^{N} D_k(x)$, where $\{D_k(x)\}$ are the Dirichlet kernels. Recall that we saw in class that the Fejer kernel can also be written as follows:

$$F_N(x) = \frac{1}{N+1}\left(\frac{\sin((N+1)\pi x)}{\sin(\pi x)}\right)^2.$$

(a) Compute the Fourier coefficients of F_N.

(b) Show that for every $j \in \mathbb{Z}$, $\lim_{N\to\infty} \widehat{F_N}(j) = 1$.

(c) Prove that $\lim_{N\to\infty} \|F_N\|_2 = \infty$.

4. Let $f \in L^1(\mathbb{T})$ such that $\{\widehat{f}(n)\}_{n\in\mathbb{Z}} \in \ell^1(\mathbb{Z})$. Show that there is a countinuous function g such that $g = f$ a.e.

5. Riemann-Lebesgue Lemma says that the Fourier coefficients of a function in $L^1(\mathbb{T})$ tend to zero, but they can tend arbitrarily slowly. Given a sequence of positive numbers $\{\epsilon_n\}_{n\in\mathbb{N}}$, prove that there is a function $f \in C[0,1]$ such that $|\widehat{f}(n)| + |\widehat{f}(-n)| \geq \epsilon_n$ for infinite values of n (hint: consider an appropriate subsequence of $\{\epsilon_n\}$ and use problem 4).

6. Show that if $|x| < \frac{|A|}{2\pi}$, then $\lim_{R\to\infty} \int_{-R}^{R} \frac{\sin(Ay)}{y} e^{2\pi i x y}\, dy = \pi$.

7. Multipliers for the Fourier transform. Let $m = \{m_n\}_{n\in\mathbb{Z}}$ be a sequence of numbers. For $f \in L^2(\mathbb{T})$, we define an operator T by

$$T_m f(x) = \sum_{n\in\mathbb{Z}} m_n \widehat{f}(n) e^{2\pi i n x}.$$

(a) Show that if $m \in \ell^\infty(\mathbb{Z})$, then T_m is a bounded linear operator from $L^2(\mathbb{T})$ to $L^2(\mathbb{T})$.

(b) Conversely, if $T_m f \in L^2(\mathbb{T}$ for every $f \in L^2(\mathbb{T})$, then $m \in \ell^\infty(\mathbb{Z})$.