Analysis Qualifying Exam
August 2019

Submit six of the problems from part 1, and three of the problems from part 2. Start every problem on a new page, label your pages and write your student ID on each page.

1 Real Analysis

Throughout, m denotes Lebesgue measure. You may use without proof the Lebesgue dominated convergence theorem, monotone convergence theorem, and Fubini's theorem.

1. For a set $M \subseteq \mathbb{R}^n$ let $\rho_M(x) = \inf_{m \in M} \|x - m\|$ where $\|.\|$ is the Euclidean distance function. Show that for two closed, disjoint sets M, N the function
$$f(x) = \frac{\rho_M(x)}{\rho_M(x) + \rho_N(x)}$$
is continuous on $\mathbb{R}^n$, satisfies $f(x) = 1$ for $x \in N$ and $f(x) = 0$ for $x \in M$.

2. Let f_n be real-valued Borel measurable functions. Show that g defined by $g(x) = \sup_n f_n(x)$ is Borel measurable.

3. Suppose $f_n : X \to [0, \infty]$ is a sequence of measurable functions with $f_1 \geq f_2 \geq ... \geq 0$, $f_n \to f$ pointwise, and $f_1 \in L^1(X, \mu)$. Prove that
$$\lim_{n\to\infty} \int_X f_n d\mu = \int_X f d\mu.$$
Show that the conclusion fails if $f_1 \notin L^1(X, \mu)$.

4. Let $\mu = \mu_F$ be the signed measure on $\mathbb{R}$ with distribution function
$$F(x) = \begin{cases} -e^x & \text{if } x < 0, \\ e^{-x} & \text{if } x \geq 0. \end{cases}$$
Find the total variation measure of μ.

5. Let X be an uncountable set and $\mathcal{A}$ the collection of all sets $E \subseteq X$ such that either E or E^c (complement) is at most countable. In the first case set $\mu(E) = 0$, in the second case set $\mu(E) = 1$. Show that $\mathcal{A}$ is a sigma-algebra and μ a measure on $\mathcal{A}$.

6. Show that if f, g are integrable in $\mathbb{R}^n$, then $f(x - y)g(y)$ is integrable in $\mathbb{R}^{2n}$.

7. Prove that if f is integrable on $\mathbb{R}^n$, real-valued and $\int_E f(x)dx \geq 0$ for every measurable set E, then $f(x) \geq 0$ a.e.x.

8. Let $f \in L^1(\mathbb{R})$ and let $t \in \mathbb{R}$. Prove that $f(tx)$ converges to $f(x)$ in the L^1-norm as $t \to 1$.

9. Let F be continuous on $[a, b]$. Assume that $F'(x)$ exists for every $x \in (a, b)$, and that $|F'(x)| \leq M$. Prove that F is absolutely continuous.

2 Complex, Functional, and Harmonic Analysis

10. Use the residue theorem to evaluate

$$\int_{\mathbb{R}} e^{5ix} \frac{\sin x}{x} dx.$$

(Hint: $\sin x = 1/(2i)(e^{ix} - e^{-ix})$ and contour deformation.)

11. Let $f : \{z \in \mathbb{C} : |z| < 1\} \to \mathbb{R}$ be analytic. Show that f is constant.

12. Assume f is entire, and there exists $C > 0$ so that

$$|f(z)| \leq C(|z| + 1)^{5/2}$$

for all complex z. Show that f is a polynomial. What is the largest degree that f may have?

13. Let f be integrable in $\mathbb{R}^n$ and not identically zero. Let $Mf(x) = \sup_B \int_B |f(y)|dy$ be the Hardy-Littlewood maximal function of f, where the supremum is taken over all balls B containing x.

    (a) Show that $Mf(x) \geq \frac{c}{|x|^n}$ for some $c > 0$ and all $|x| \geq 1$.

    (b) Is $Mf(x)$ integrable in $\mathbb{R}^n$? Justify your answer.

14. Show that the Hardy-Littlewood maximal operator (see previous problem) is bounded from L^1 to $weak L^1$. You can do it in dimension $n = 1$.

15. Let $\star$ be the convolution operation, defined by $f \star g(x) = \int_{\mathbb{R}^n} f(x - y)g(y)dy$. Use the Fourier transform to show that there is no function $g \in L^1(\mathbb{R}^n)$ with the property that $f \star g = f$ for all $f \in L^1(\mathbb{R}^n)$.

16. Let X and Y be normed spaces. Prove that if Y is Banach space, then so is $B(X, Y)$.

17. Let H be a Hilbert space.

    (a) Define the *absolute* $|T|$ of an operator $T \in B(H)$.

    (b) Prove that if $T \in B(H)$ with $|T| = |T^*| = I$, then T is unitary.

18. (a) State the Open Mapping Theorem.

    (b) Let X and Y be Banach spaces and $T \in B(X, Y)$. Prove that if T is 1-1 and onto, then it is bounded below (i.e., $\inf_{\|x\|=1} \|Tx\| > 0$).