Analysis Preliminary Exam
Measure Theory and Integration

Submit six of the following problems. Start every problem on a new page, label your pages, and write your student ID on each page.

1. Let $\mu_F$ be the Lebesgue-Stieltjes measure associated to the function

$$F(x) = \begin{cases} 
0, & x \leq 0 \\
1, & x > 0.
\end{cases}$$

(a) Describe the largest $\sigma$-algebra on the real line on which the measure $\mu_F$ is defined.

(b) Is $\mu_F$ absolutely continuous with respect to the Lebesgue measure $m$? Justify your answer.

2. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous function on all points in $\mathbb{R}^n \setminus A$, where $A$ is a set with zero Lebesgue measure. Prove that $f$ is a measurable function.

3. Compute the following integral, and justify your answer using appropriate convergence theorems.

$$\lim_{n \to \infty} \int_0^1 \frac{\cos nx}{1 + n \sqrt{x}} dx.$$ 

4. Study whether the function $f(x, y) = \frac{1}{x^2 + y^2}$ is Lebesgue integrable in $\mathbb{R}^2$. Justify your answer.

5. Give an example of a sequence $\{f_n\}_{n=1}^\infty$ of non-negative functions on the interval $[0, 1]$ that satisfies the following properties:

   (i) $\lim_{n \to \infty} \int_0^1 f_n(x) dx = 0$.
   (ii) $f_n$ is continuous for all $n \geq 1$.
   (iii) The sequence $\{f_n(x)\}_{n=1}^\infty$ does not converge for any $x \in [0, 1]$.

6. Answer the following questions, justifying your answers (give a counterexample or a proof, as needed): 

   (a) If a sequence of functions converges in norm (in the integral) does this imply that the sequence converges pointwise?
   
   (b) If a sequence of functions converges in norm (in the integral) does this imply that a subsequence of the given sequence converges pointwise?

7. Let $(X, \mu)$ be a (not necessarily finite) measure space. Assume that $f, f_n : X \to [0, \infty]$ are non-negative functions such that $f_n$ converges to $f$ almost everywhere, $\int_X f d\mu < +\infty$, $\int_X f^4 d\mu < +\infty$, $\int_X f_n d\mu \to \int_X f d\mu$, and $\int_X f_n^4 d\mu \to \int_X f^4 d\mu$. Prove that $\int_X f_n^2 d\mu \to \int_X f^2 d\mu$.

   Hint: Consider the subsets of $X$ where $f$ is greater than 1 or less than 1.

8. Let $\{f_n\}$ be a sequence of absolutely continuous functions on $[0, 1]$, such that $\{f_n\}$ converges to a $f$ in $L^1$ and $\{f_n\}$ is a Cauchy sequence in $L^1$. Prove that there exists an absolutely continuous function $\tilde{f}$ on $[0, 1]$ such that $\tilde{f} = f$ almost everywhere.