Analysis Preliminary Exam
Measure Theory and Integration

Submit six of the following problems. Start every problem on a new page, label your pages, and write your student ID on each page.

1. Let μ_F be the Lebesgue-Stieltjes measure associated to the function
$$F(x) = \begin{cases} 0, & x \leq 0 \\ 1, & x > 0. \end{cases}$$
   (a) Describe the largest σ-algebra on the real line on which the measure μ_F is defined.
   (b) Is μ_F absolutely continuous with respect to the Lebesgue measure m? Justify your answer.

2. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous function on all points in $\mathbb{R}^n \setminus A$, where A is a set with zero Lebesgue measure. Prove that f is a measurable function.

3. Compute the following integral, and justify your answer using appropriate convergence theorems.
$$\lim_{n\to\infty} \int_0^1 \frac{\cos nx}{1 + n\sqrt{x}} dx.$$

4. Study whether the function $f(x, y) = \frac{1}{x^2+y^2}$ is Lebesgue integrable in $\mathbb{R}^2$. Justify your answer.

5. Give an example of a sequence $\{f_n\}_{n=1}^\infty$ of non-negative functions on the interval $[0, 1]$ that satisfies the following properties:
   (i) $\lim_{n\to\infty} \int_0^1 f_n(x)\, dx = 0$.
   (ii) f_n is continuous for all $n \geq 1$.
   (iii) The sequence $\{f_n(x)\}_{n=1}^\infty$ does not converge for any $x \in [0, 1]$.

6. Answer the following questions, justifying your answers (give a counterexample or a proof, as needed):
   (a) If a sequence of functions converges in norm (in the integral) does this imply that the sequence converges pointwise?
   (b) If a sequence of functions converges in norm (in the integral) does this imply that a subsequence of the given sequence converges pointwise?

7. Let (X, μ) be a (not necessarily finite) measure space. Assume that $f, f_n : X \to [0, \infty)$ are non-negative functions such that f_n converges to f almost everywhere, $\int_X f d\mu < +\infty$, $\int_X f^4 d\mu < +\infty$, $\int_X f_n d\mu \to \int_X f d\mu$, and $\int_X f_n^4 d\mu \to \int_X f^4 d\mu$. Prove that $\int_X f_n^2 d\mu \to \int_X f^2 d\mu$.
   Hint: Consider the subsets of X where f is greater than 1 or less than 1.

8. Let $\{f_n\}$ be a sequence of absolutely continuous functions on $[0, 1]$, such that $\{f_n\}$ converges to a f in L^1 and $\{f_n'\}$ is a Cauchy sequence in L^1. Prove that there exists an absolutely continuous function $\widetilde{f}$ on $[0, 1]$ such that $\widetilde{f} = f$ almost everywhere.