Analysis Preliminary Exam
January 16, 2010

1. Let $(X, \mathcal{M})$ be an abstract measure space with $X = A \cup B$, where $A, B \in \mathcal{M}$. Show that a function f on X is measurable if and only if f is measurable on A and f is measurable on B.

2. Show using the definition of Lebesgue outer measure that if $X \subseteq \mathbb{R}$ is countable then X is measurable and $m^*(X) = 0$.

3. Let f be a bounded real valued function on $[a, b]$. Prove that if f is Riemann integrable, then f is Lebesgue integrable and
$$\int_a^b f(x)dx = \int_{[a,b]} f dm.$$
Hint: Recall the definition of Riemann integral: For every partition $P = \{t_j\}_{j=0}^n$, such that $a = t_0 < t_1 < \cdots < t_n = b$ of $[a, b]$, define
$$S_P f = \sum_{j=1}^n M_j(t_j - t_{j-1}) \quad \text{and} \quad s_P f = \sum_{j=1}^n m_j(t_j - t_{j-1}),$$
where M_j and m_j are the supremum and infimum of f on $[t_{j-1}, t_j]$. If $\inf_P S_P f = \sup_P s_P f$, then their common value is the Riemann integral of f, denoted by $\int_a^b f(x)dx$.

4. Justify the identity $\displaystyle\int_0^1 \sum_{n=0}^\infty (-x)^n \, dx = \sum_{n=0}^\infty \int_0^1 (-x)^n \, dx$, and use it to prove that $\displaystyle\sum_{n=0}^\infty \frac{(-1)^n}{n+1} = \ln 2.$

5. Let g be a measurable function. Assume that for every $f \in L^1(\mathbb{R})$, we have $gf \in L^1(\mathbb{R})$. Show that $g \in L^\infty(\mathbb{R})$.

6. Let $(X, \mathcal{M}, \mu)$ be a measure space with $\mu(X) < \infty$. Assume that $\displaystyle\int_X f \, d\mu = \int_X g \, d\mu$. Prove that either $f = g$ a.e., or there exists a set $E \in \mathcal{M}$ with $\mu(E) > 0$ such that $\displaystyle\int_E f \, d\mu > \int_E g \, d\mu$.