Analysis Preliminary Examination
August 2016

Submit six of the problems from part 1, and three of the problems from part 2. Start every problem on a new page, label your pages and write your student ID on each page.

Part 1: Real Analysis

Lebesgue measure is denoted m and unless stated otherwise $(X, \mathcal{M}, \mu)$ is a generic measure space.

1. Let A be an algebra on X. Prove that A is a σ-algebra if and only if it is closed for countable increasing sequences (*i.e.*, if $E_i \in A$ and $E_1 \subseteq E_2 \subseteq E_3 \subseteq \cdots$, then $\cup_{n=1}^{\infty} E_i \in A$).

2. Let μ^* be an outer measure, let H be a μ^*-measurable set, and let μ_0^* be the restriction of μ^* to $\mathcal{P}(H)$.

   (a) Check that μ_0^* is an outer measure on H.

   (b) Prove that $A \subseteq H$ is μ_0^*-measurable if and only if A is μ^*-measurable.

   (c) If H is not assumed to be μ^*-measurable, do (a) and (b) still hold?

3. Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Prove or give a counterexample for each one of the following statements.

   (a) If f is a (Lebesgue) measurable function, then f^+ and f^- are measurable.

   (b) If f and $|f|$ are measurable, then f^+ and f^- are measurable.

   (c) If f is not measurable, then neither f^+ nor f^- are measurable.

4. Let $(X, \mathcal{M}, \mu)$ be a measure space with $\mu(X) < \infty$, and let $\{f_n\}_{n=1}^{\infty}$ be a sequence of measurable functions. Prove that if f_n converges to 0 almost everywhere, then f_n converges to 0 in measure. Show with an example that the hypothesis $\mu(X) < \infty$ is necessary.

5. Let $f_n : [1, \infty) \to \mathbb{R}$ be defined by

$$f_n(x) = \frac{1}{x}\chi_{[n,\infty)}(x).$$

   Study the pointwise, uniform, and L^1 convergence of the sequence $\{f_n\}$. Specify the theorems you use.

6. Check that the hypotheses of Fubini's theorem hold, and use the theorem to compute the following integral

$$\int_0^{\pi/2} \int_y^{\pi/2} y \frac{\sin x}{x} dx dy.$$

7. A function $f : \mathbb{R} \to \mathbb{R}$ is called Lipschitz if there exists a constant $C > 0$ such that, for every $x, y \in \mathbb{R}$, $|f(x) - f(y)| \leq C|x - y|$.

(a) Prove that every Lipschitz function is absolutely continuous.

(b) Give an example of an absolutely continuous function that is not Lipschitz.

8. Let $\{a_n\}$ and $\{b_n\}$ be two sequences of positive real numbers such that $\inf a_n = 0$ and $\inf b_n > 0$. Let μ, ν be the measures on $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$ defined by $\mu(n) = a_n$, $\nu(n) = b_n$.

   (a) Show that μ is absolutely continuous with respect to ν, and that the hypotheses of the Radon-Nikodym Theorem are satisfied.

   (b) Find $\frac{d\mu}{d\nu}$.

Part 2: Complex and Functional Analysis

9. Let $(X, \mathcal{M}, \mu)$ be a finite measure space and consider the associated $L_p(X, \mathcal{M}, \mu)$-space.

   (i) State Minkowski's Inequality in $L_p,\ 1 \leq p < \infty$.

   (ii) Using Hölder's Inequality, Prove Minkowski's Inequality in $L_p,\ 1 \leq p < \infty$.

10. Let $(X, \|\ \|_1)$ and $(Y, \|\ \|_2)$ be normed spaces and let $B(X, Y)$ denote the space of bounded linear operators from X to Y with the usual norm $\|T\| = \sup_{\|x\|_1} \|Tx\|_2$. Prove that $B(X, Y)$ is a Banach space if $(Y, \|\ \|_2)$ is a Banach space.

11. Let H be a Hilbert space and $M \subset H$ be a closed subspace.

    (i) Define the orthogonal complement $M^\perp$ of M (in H).

    (ii) Prove that $M = M^{\perp\perp}$.

12. Let X be a complete metric space and $S \subset X$ be a dense set. Prove that if $S = \bigcap_n A_n$, where each $A_n \subset X$ is open, then $X \setminus S$ is of first category.

13. (a) Let G be the open unit disk. If $u : G \to \mathbb{R}$ is harmonic, show that u has a harmonic conjugate.

    (b) Find an example that illustrates that the statement is false if G is the punctured unit disk.

14. Let G be a region and suppose that $f : G \to \mathbb{C}$ is analytic such that $f(G)$ is a subset of a circle. Show that f is constant.

15. Give the power series expansion of log(z) about z=i and determine its radius of convergence.

16. Let G be a region, and assume that f and g are two analytic functions on G with $f(z)g(z) = 0$ for all $z \in G$. Show that either f or g is the zero function.