Analysis Preliminary Examination - September 2013

Part 1: Measure Theory

Provide solutions to 6 of the 10 problems in the first part.

I.1 Let (X, M, μ) be a measure space and $f : X \to \mathbb{R}$ a measurable function, finite at every $x \in X$. Let $G_f = \{(x,t) \in X \times \mathbb{R} : t = f(x)\}$ be the graph of f. If μ is σ-finite, prove that G_f has measure zero with the product measure $\mu \times m$ (Hint: use Fubini).

I.2 Let $X = [0,1]$, $M = B_{[0,1]}$, m the Lebesgue measure on M and μ the counting measure on M. Show that $m \ll \mu$, but that there is no function f such that $dm = f d\mu$. Does this contradict the Radon-Nikodym theorem?

I.3 (a) Let $f_n : [1,\infty) \to \mathbb{R}$ be a function defined by $f_n(x) = \frac{1}{x}\chi_{[n,\infty)}(x)$. Show that the sequence $\{f_n\}$ converges to zero uniformly on $[1,\infty)$.

(b) State Fatou's Lemma.

(c) Apply Fatou's lemma to the sequence from part (a).

I.4 Let $f_n : X \to \mathbb{R}$ be measurable, bounded functions, such that for every $n \in \mathbb{R}, x \in X$, $f_n(x) \geq f_{n+1}(x)$, and there is a measurable function $f : X \to \mathbb{R}$ such that $\lim f_n(x) = f(x)$ pointwise. If $\int f_k d\mu < \infty$ for some $k \in \mathbb{N}$, prove that

$$\lim_{n\to\infty} \int f_n d\mu = \int f d\mu.$$

I.5 Prove the following:

(a) If f is monotonic, then f is Lebesgue measurable.

(b) If f is continuous and g is Lebesgue measurable, then $f \circ g$ is Lebesgue measurable.

(c) If f is continuous and g is Lebesgue measurable, is $g \circ f$ Lebesgue measurable? Justify your answer.

I.6 Let $f(x) = \int_0^\infty e^{-xt}\left(t^{-3}\sin^3 t\right) dt$. Show that

(a) $f(x)$ is well-defined for each $x \in [0,\infty)$.

(b) $f(x)$ is continuous on $[0,\infty)$.

I.7 Suppose that f is a continuous real-valued function of bounded variation on $[0,1]$ and that, for each $\varepsilon \in (0,1)$, f is absolutely continuous on $[\varepsilon, 1]$. Must f necessarily be absolutely continuous on $[0,1]$? Justify your answer.

I.8 Suppose $f \in L^1[0,1]$ satisfies

$$\int_E |f| \leq (m(E))^2$$

for every measurable set $E \subseteq [0,1]$. Show that f is a.e. equal to zero. (Hint: Lebesgue differentiation theorem.)

I.9 Prove or give a counterexample: Every dense open subset of $(0, 1)$ has Lebesgue measure one.

I.10 Let $f \in L^1(\mathbb{R})$ such that $f(x) = 0$ for $|x| \geq 1$ (Is this condition necessary?) Prove that f_n defined by

$$f_n(x) = f\left(x + \frac{1}{n}\right)$$

converges to f in $L^1(\mathbb{R})$.

Part 2:Complex and Functional Analysis

Provide solutions to 3 of the 6 problems of part 2.

II.1 Let $G \subseteq \mathbb{C}$ be a region. If $f : G \to \mathbb{C}$ is analytic except for poles, show that the poles of f cannot have a limit point in G.

II.2 An analytic function has a singularity (removable, pole, essential) at infinity, if $f(1/z)$ has the same type of singularity at the origin. Show that an entire function that has a removable singularity at infinity is constant.

II.3 Prove that there is no branch of the logarithm defined on $\mathbb{C}\backslash\{0\}$.

II.4 A normed space $(X, \|\cdot\|)$ is called *strictly convex* if whenever $x, y \in X$ verify

$$\|x\| = \|y\| = \frac{1}{2}\|x + y\|,$$

then it follows that $x = y$.

(a) Prove that a Hilbert space is always strictly convex.

(b) Give an example of a Banach space that is not strictly convex (Hint: It is enough to consider $\mathbb{R}^2$ with an appropriate norm).

II.5 Let X, Z be Banach spaces and Y a normed space over $\mathbb{R}$. Let $T : X \to Y$ be a bounded linear operator, and $S : Y \to Z$ a closed linear operator. Prove that $S \circ T$ is a bounded operator.

II.6 Let M be a closed subspace of the Banach space X. Let $x_0 \in X$ be such that the distance from x_0 to M is positive (the distance is defined by $d(x_0, M) = \inf\{\|x_0 - y\| : y \in M\}$). Prove that there exists a functional $F \in X^*$ with the following properties:

(a) $F(x) = 0$ for every $x \in M$.

(b) $F(x_0) = d(x_0, M)$.

(c) $\|F\| = 1$