Analysis Qualifying Exam, January 2015

Submit six of the problems from part 1, and three of the problems from part 2. Start every problem on a new page, label your pages and write your student ID on each page.

Part 1 - Real Analysis

Lebesgue measure is denoted by m.

1. a) State the definition of Lebesgue outer measure.

b) Prove that Lebesgue outer measure is translation invariant; i.e., $m^*(A) = m^*(\alpha + A)$, $\forall \alpha \in \mathbb{R}$ and $\forall A \subset \mathbb{R}$ measurable.

2. Let $A = \mathbb{Q}^c \cap [0,1]$.

a) Is A Lebesgue measurable? If so, what is $m(A)$?

b) Is there a closed set F of positive measure such that $F \subset A$? Justify your answer.

3. a) State the definition of a Lebesgue measurable function.

b) For a function $f : \mathbb{R} \to \mathbb{R}$ which of the following is true (justify your answers): (i) if $|f|$ is measurable, then f is also measurable. (ii) if f is measurable, then $|f|$ is also measurable.

4. a) State Chebychev's Inequality.

b) Let $f : [0,1] \to \mathbb{R}$ be a measurable function. Prove that $\int_{[0,1]} f dm = 0$ iff $f = 0$ a.e. on $[0,1]$.

5. a) State the definition of a function of bounded variation $f : [a,b] \to \mathbb{R}$.

b) Let $f, g : [a,b] \to \mathbb{R}$ be two functions of bounded variation. Is fg a function of bounded variation? How about the function $|f|$? Justify your answers.

6. a) Give an example of a function of bounded variation which is not absolutely continuous.

b) Let $f : [0,1] \to \mathbb{R}$ be an integrable function and define $F : [0,1] \to \mathbb{R}$ by $F(x) = \int_{[0,x]} f(t) dm$ for all $x \in [0,1]$. Prove that F is absolutely continuous on $[0,1]$.

7. Let $(X, \mathcal{A}, \mu)$ be a measure space.

a) If $f : X \to \mathbb{R}$ is a non-negative $\mathcal{A}$-measurable function, show that the set function $\nu(E) := \int_E f d\mu$ is a measure on $(X, \mathcal{A})$.

b) Is $\nu << \mu$? Justify your answer.

c) If $g : X \to \mathbb{R}$ is a $\mathcal{A}$-measurable function, show that

$$\int_X g d\nu = \int_X f g d\mu.$$

8. Consider the measure spaces $([0,1], \mathcal{F}, m)$ and $([0,1], \mathcal{P}([0,1]), c)$, where c is the counting measure. For $E = \{(x,y) : x = y\} \in \mathcal{F} \times \mathcal{P}([0,1])$, calculate the iterated integrals

$$\int_{[0,1]} [\int_{[0,1]} \chi_E(x,y) dc] dm \quad \text{and} \quad \int_{[0,1]} [\int_{[0,1]} \chi_E(x,y) dm] dc$$

and the double integral

$$\int_{[0,1]\times[0,1]} \chi_E(x,y) d(m \times c).$$

Do your answers contradict Fubini-Tonelli Theorem? Explain.

Part 2 - Complex and Functional Analysis

1. Let u and v be real valued harmonic functions on a domain Ω. If u and v agree on a set with a limit point in Ω, does it follow that $u = v$ on all of Ω? Explain.

2. Let f, g be analytic functions in a neighborhood of a point a. Assume that g has a simple zero at a. Find a formula for the residue of $\frac{f(z)}{g(z)^2}$ at a in terms of $f(a)$ and the derivatives of f and g at a.

3. Let $\log z$ denote the branch of a complex logarithm with branch cut along the negative imaginary axis such that $-\frac{\pi}{2} < Im \log z < \frac{3\pi}{2}$. Let C_r denote the curve parametrized by $z(t) = re^{it}$, $0 \le t \le \pi$. Prove that

$$\int_{C_r} \frac{\log z}{(z^2+1)^2} dz$$

tends to zero as r tends to infinity AND as r tends to zero. Use this to compute

$$\int_0^\infty \frac{\log x}{(x^2+1)^2} dx.$$

4. Find a sequence of one-to-one conformal mappings from the half disk $\{z \in \mathbb{C} : |z| \le 1, Im z \ge 0\}$ to the unit disk $\{z \in \mathbb{C} : |z| \le 1\}$.

5. Let H be a Hilbert space. Let $x \in H$ such that $\|x\| \le 1$, and let $\{x_n\}$ be a sequence in H such that $\|x_n\| \le 1$ for every n. Assume that $\lim_{n\to\infty} \|x_n - x\| = 2$. Is it true that x_n must converge to x? Justify your answer.

6. Let X be a normed space, and T be a bounded operator on X. Prove that

$$\|T\| = \sup\{|f(T(x))| : x \in X, ||x|| = 1; f \in X^*, \|f\| = 1\}.$$

7. Let $s \in (0, 3/4)$ be a fixed real number. For $f \in L^4(0,1)$ and $x \in (0,1)$, we define the operator T by

$$Tf(x) = \int_0^x \frac{f(y)}{(x-y)^s} dy.$$

Prove that T is a bounded operator from $L^4(0,1)$ to $L^\infty(0,1)$.

8. Let X, Y, Z be three Banach spaces over the same scalar field. Let $T : X \to Y$ be a bounded operator, and $S : Y \to Z$ be a closed linear operator. Prove that $S \circ T$ is a bounded operator from X to Z.