Analysis Preliminary Examination
January 2011

Unless a problem states otherwise, $\mathbb{R}$ is endowed with Lebesgue measure, which will be denoted by m. Please justify your answers.

1. a) Give the definition of the (Lebesgue) outer measure $m^*(A)$ of a set $A \subset \mathbb{R}$.

   b) Prove that $m^*(\mathbb{Q}) = 0$. Is $\mathbb{Q}$ is measurable? Why?

2. A set function μ on a σ-algebra $\mathcal{A}$ of subsets of a set X is called **continuous from above** if $\{E_n\} \subset \mathcal{A}$ with $E_n \downarrow$, then $\mu(\cup_n) = \lim_n \mu(E_n)$. Let μ be a finitely additive measure on a measurable space $(X, \mathcal{A})$. When $\mu(X) < \infty$, show that μ is a measure iff it is continuous from above.

3. a) Give the defintion of a measurable function $f : E \to \mathbb{R}$, where E is a measurable subset of $\mathbb{R}$.

   b) Prove that if f and g are measurable on E, then so is $f + g$.

4. a) State the Bounded Convergence Theorem for a sequence $\{f_n\}$ of measurable functions on a measurable set $E \subset \mathbb{R}$.

   b) Show, by an example, that the boundedness assumption is necessary.

5. a) Give the definition of a function of bounded variation on the interval $[a,\, b]$.

   b) Show that if f and g are functions of bounded variation on $[a,\, b]$, then their product $f \cdot g$ is also of bounded variation.

6. Consider the measure spaces $([0,1], \mathcal{F}, m)$ and $([0,1], \mathcal{P}([0,1]), \mu)$, where μ is the counting measure on $\mathcal{P}([0,1])$. Let $D = \{(x,x) : x \in [0,1]\}$ and $f = \chi_D$ on $[0,1] \times [0,1]$. Show that
$$\int (\int f_x d\mu) dm \neq \int (\int f^x dm) d\mu.$$
   Does this contradict Fubini's Theorem?

7. Let $f \in L^p \cap L^q$, where $0 < p < q < +\infty$. If $p < r < q$, show that $f \in L^r$ and that $\varphi : [p,q] \to \mathbb{R}$, defined by $\varphi(r) = \ln(\|f\|_r^r)$, is convex.

8. Let $A \in \mathcal{F}$ and $\{f_n\}$ be a sequence of measurable functions such that $\forall n \in \mathbb{N},\ \forall x \in A,\ |f_n(x)| \leq \frac{1}{n^2}$. Let g be an integrable function on A. Show that
$$\int_A (\sum_{n=1}^{\infty} f_n g) dm = \sum_{n=1}^{\infty} \int_A f_n g dm.$$

9. Let $f,\ f_n,\ n = 1,\ 2,\ \ldots$ be Lebesgue measurable functions defined on $\mathbb{R}$ such that $|f_n(x) \leq \frac{1}{|x|}$ a.e. on $\mathbb{R}$, for each $n \in \mathbb{N}$, and $\lim_{n\to\infty} f_n(x) = f(x)$ a.e. on $\mathbb{R}$. Show that $f_n \to f$ in measure.

10. Let f and g be integrable functions on $[a, b]$ and let $F(x) = \alpha + \int_a^x f(t)dt,\ G(x) = \beta + \int_a^x g(t)dt \ \ \forall x \in [a, b]$. Show that

$$\int_a^b G(t)f(t)dt + \int_a^b g(t)F(t)dt = F(b)G(b) - F(a)G(a).$$