Analysis Preliminary Examination
January 2017

Submit six of the problems from part 1, and three of the problems from part 2. Start every problem on a new page, label your pages and write your student ID on each page.

Part 1: Real Analysis

Lebesgue measure is denoted m and unless stated otherwise $(X, \mathcal{M}, \mu)$ is a generic measure space.

1. Let μ_1^* and μ_2^* be two outer measures defined on $\mathcal{P}(\mathcal{X})$, such that $\mu_1^*(X) < \infty$ and $\mu_2^*(X) < \infty$. Let $\mathcal{M}_\infty$ and $\mathcal{M}_\in$ be the σ-algebra of the measurable sets for μ_1^* and μ_2^*, respectively. We define
$$\mu^*(A) = \mu_1^*(A) + \mu_2^*(A), \quad \forall A \subseteq X.$$
   (a) Show that μ^* is an outer measure.
   (b) Describe the σ-algebra of measurable sets for μ^*.

2. Given a set $E \subseteq \mathbb{R}$, we define a measure μ by
$$\mu(E) = \sum_{n=1}^{\infty} \chi_E(1/n).$$
   (a) Show that μ is a measure defined on $\mathcal{P}(\mathbb{R})$, that is σ-finite but not finite.
   (b) Explain why μ does not coincide with any Lebesgue-Stieltjes measure dF.
   (c) Compute the integral $\int_R f d\mu$ for $f(x) = e^{-1/x}\chi_(0, \infty)$.

3. Let $f_n, f : X \to \mathbb{R}$ be measurable functions.
   (a) Give the definition of convergence of f_n to f in the L^1-norm, and in measure.
   (b) Prove that if f_n converges to f in the L^1-norm, then f_n converges to f in measure.
   (c) Give an example of a sequence $\{f_n\}$ and a function f such that f_n converges to f in measure, but not in the L^1-norm.

4. Let $f_n(x) = \frac{n}{1+n^2x^2}$, for $n \in \mathbb{N}$. Show that f_n is pointwise convergent for every $x > 0$, but that
$$\lim_{n\to\infty} \int_0^\infty f_n(x)dx \neq \int_0^\infty \lim_{n\to\infty} f_n(x)dx.$$
   Which hypothesis of the Dominated Convergence Theorem is not satisfied in this case?

5. Let I_k denote the interval $\left(\frac{1}{k+1}, \frac{1}{k}\right]$, and let $g_k(x) = k(k+1)\chi_{I_k}(x)$. We define $f : (0, 1] \times (0, 1] \to \mathbb{R}$ by
$$f(x, y) = \sum_{k=1}^{\infty} (g_k(x) - g_{k+1}(x))\, g_k(y).$$

Show that the two iterated integrals $\int_0^1 \int_0^1 f(x,y)dxdy$ and $\int_0^1 \int_0^1 f(x,y)dydx$ exist and have different values. Explain why this does not contradict Fubini's theorem.

6. Let ν be a signed measure on $(X, \mathcal{M})$.

   (a) Give the definition of positive, negative and null sets for ν.

   (b) Prove that a countable union of positive sets for ν is a positive set for ν.

7. Let dF be the Lebesgue-Stieltjes measure associated to the function

$$F(x) = \begin{cases} 0 & x < 0, \\ x^3, & 0 \leq x < 1, \\ 1, & 1 \leq x. \end{cases}$$

   Show that dF is absolutely continuous with respect to Lebesgue measure, and find the Radon-Nikodym derivative dF/dm.

8. Let δ_0 be the delta measure at zero (*i.e.* $\delta_0(A) = 1$ if $0 \in A$, and $\delta_0(A) = 0$ otherwise). Let $g_h(x) = \frac{1}{2h}\chi_{[-h,h]}(x)$, and we define a measure ν_h by

$$\nu_h(A) = \int_A g_h(x)dx.$$

   Show that for every f continuous we have

$$\lim_{h \to 0} \int_{\mathbb{R}} f d\nu_h = \int_{\mathbb{R}} f d\delta_0.$$

Part 2: Complex and Functional Analysis

9. Let c be the collection of all convergent sequences of real numbers endowed with the norm $\|(a_n)\|_\infty = \sup_n |a_n|$..

   a) State the definition of a separable normed space.

   b) Show that $(c, \| \|_\infty)$ is separable.

10. Consider $C[0,1]$ with norms $\| \|_\infty$ and $\| \|_1$. Define a linear functional $p : C[0,1] \to \mathbb{R}$ by $p(f) = f(0)$.

   a) Show that p is continuous on $(C[0,1], \| \|_\infty)$.

   b) Show that p is not continuous on $(C[0,1], \| \|_1)$.

11. Let M and N be closed subspaces of a Hilbert space H. Prove that $M \oplus N$ is closed provided that $x \perp y$ for all $x \in M,\ y \in N$.

12. Let $\mathcal{H}$ denote the set of Hermitian operators on a Hilbert space H. Prove that

   a) $\mathcal{H}$ is a real vector subspace of $B(H)$.

b) If $T \in B(H)$, then the operators $T + T^*, T^*T, TT^*$ are all in $\mathcal{H}$.

13. Let u and v be real valued harmonic functions on a domain Ω. If u and v agree on a set with a limit point in Ω, does it follow that $u = v$ on all of Ω? Explain.

14. Let f be an entire function mapping $\mathbb{C}$ into the unit disk. Show that f is constant.

15. Give the power series expansion of log(z) about z=i and determine its radius of convergence.

16. Show that
$$\int_{-\infty}^{\infty} \frac{\sin x}{x} dx = \pi.$$
(Hint: Deform the contour and use $\sin x = \frac{1}{2i}(e^{ix} - e^{-ix})$.)