Analysis Preliminary Examination
June 2011

Unless a problem states otherwise, $\mathbb{R}$ is endowed with Lebesgue measure, which will be denoted by m. Please justify your answers.

1. a) Give the definition of the (Lebesgue) outer measure $m^*(A)$ of a set $A \subset \mathbb{R}$.

   b) Prove that if $A \subset \mathbb{R}$ is a countable set, then $m^*(A) = 0$.

   c) Prove or disprove with a counterexample: If $A \subset \mathbb{R}$ is an uncountable set, then $m^*(A) > 0$.

2. a) Show that any set $A \subset \mathbb{R}$ such that $m^*(A) = 0$ is measurable.

   b) Prove that the Lebesgue measure is translation invariant, i.e., if $A \subset \mathbb{R}$ is a measurable set and $\alpha \in \mathbb{R}$, then $m(A) = m(\alpha + A)$.

3. a) Give the defintion of a measurable function $f : E \to \mathbb{R}$, where E is a measurable subset of $\mathbb{R}$.

   b) Prove or disprove the following statements

   i) f is measurable if and only if $\{x : f(x) = \alpha\}$ is measurable for all $\alpha \in \mathbb{R}$.

   ii) f is measurable if and only if $|f|$ is measurable.

4. a) State the Monotone Convergence Theorem for a sequence $\{f_n\}$ of measurable functions on a measurable set $E \subset \mathbb{R}$.

   b) Show, by an example, that the monotonicity assumption is necessary.

5. a) State Fubini's Theorem (for product spaces).

   b) Let $(X, \mathcal{F}, \mu)$ and $(Y, \mathcal{G}, \nu)$ be measure spaces; E and F are measurable subsets of $X \times Y$ such that $\nu(E_x) = \nu(F_x)$ for almost every $x \in X$. Show that $(\mu \times \nu)(E) = (\mu \times \nu)(F)$.

6. a) State Hölder's Inequality. When does the equality hold?

   b) Let $\{f_n\} \subset L_p$ such that $\|f_n - f\|_p \to 0$, where $f \in L_p$. Show that if g is essentially bounded, then $\|f_n g - fg\|_p \to 0$.

7. Let $1 < p, q$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Assume that $\{f_n\} \subset L_p$ such that $\|f_n - f\|_p \to 0$, where $f \in L_p$ and $\{g_n\} \subset L_q$ such that $\|g_n - g\|_q \to 0$, where $g \in L_q$. Show that $\|f_n g_n - fg\|_1 \to 0$.

8. a) State the definition of absolutely continuous function on an interval $[a, b]$.

b) Let $f \in \mathcal{L}_\infty([a,b])$ and $F(x) = \int_a^x f(t)dt$, for all $x \in [a,b]$. Show that F is absolutely continuous on $[a,b]$.

9. Let $\{f_n\} \subset L_p$ and $f \in L_p$, $1 < p < \infty$, such that $f_n \to f$ a.e. and $\lim_{n\to\infty} \|f_n\|_p = \|f\|_p$. Show that $\lim_{n\to\infty} \|f_n - f\|_p = 0$.

10. Let μ, ν and λ be σ-finite measures defined on the sigma algebra $\mathcal{F}$ such that $\lambda << \mu$ and $\mu << \nu$. Show that $\lambda << \nu$, and that we have

$$\frac{d\lambda}{d\nu} = \frac{d\lambda}{d\mu}\frac{d\mu}{d\nu} \quad \nu - a.e.$$