Analysis Preliminary Examination
September 2011

- Unless a problem states otherwise m will denote Lebesgue measure.
- Please justify your answers.
- Provide solutions to 8 of the problems below (additional problems will not be graded)

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1. Let $(X, \mathcal{B}, \mu)$ be a measure space. Let $\mathcal{N} := \{N \in \mathcal{M} : \mu(N) = 0\}$ and $\overline{\mathcal{B}} := \{E \cup F : E \in \mathcal{B}$ and $F \subseteq N$ for some $N \in \mathcal{N}\}$. Prove that $\overline{\mathcal{B}}$ is a σ-algebra and there is a unique extension $\overline{\mu}$ of μ to a complete measure on $\overline{\mathcal{B}}$.
2. Let $A \subseteq \mathbb{R}$ with finite outer measure. Show that A is Lebesgue measurable if and only if there is a $B \in G_{\sigma\delta}$ with $A \subseteq B$ and $m^*(B \setminus A) = 0$.
3. If $f : \mathbb{R} \to [0, \infty]$ then f is measurable if and only if there is a sequence of simple functions φ_n such that $0 \leq \varphi_1 \leq \varphi_2 \leq \cdots \leq f$ and $\varphi_n \to f$ almost everywhere.
4. Compute the following
   (a) $\lim\limits_{n\to\infty} n(1 + n^2x^2)^{-1}$
   (b) $\lim\limits_{n\to\infty} \displaystyle\int_a^\infty n(1 + n^2x^2)^{-1}\, dx.$
   (c) Explain your answer to part b with respect to the convergence theorems.
5. Prove that if $\{f_1, f_2, \cdots, f_n\}$ are measurable functions then so is $\max\{f_1, f_2, \cdots, f_n\}$.
6. State the Bounded Convergence Theorem and show by example that boundedness is essential.
7. Answer the following:
   (a) What does it mean for a function to be of bounded variation on the interval $[a, b]$.
   (b) Describe the relationship between functions of bounded variation and monotonic functions.
   (c) Construct an example of a continuous function on $[0, 1]$ which is not of bounded variation.
8. Let $f \in L_1[0, 1]$ and $\{f_n\}_{n=1}^\infty \subseteq L_1[0, 1]$ such that $f_n \to f$ almost everywhere. Prove that $\|f_n - f\|_1 \to 0$ if and only if $\|f_n\|_1 \to \|f\|_1$.
9. Prove or provide a counterexample: If f_n converges to f in the L_p norm then $f_n \to f$ almost everywhere.
10. Let $(X, \mathcal{B}, \mu)$ be a finite measure space and $\{E_k\}_{k=1}^n \subseteq \mathcal{B}$, and $\{c_k\}_{k=1}^n$ a collection of real numbers. For $E \in \mathcal{B}$ define
$$\nu(E) = \sum_{k=1}^n c_k \mu(E_k \cap E)$$
and show that ν is absolutely continuous with respect to μ. Determine the Radon-Nikodym derivative $\frac{d\nu}{d\mu}$.