Analysis Preliminary Examination
January 2013

- Unless a problem states otherwise m will denote Lebesgue measure, m^* will denote Lebesgue outer measure, and $\mathcal{L}$ will denote the Lebesgue measurable sets. You may use without proof the Lebesgue dominated convergence theorem and the Fubini-Tonelli Theorem.
- Please justify your answers.
- Clearly indicate which problems you wish to be considered for grading.

Part I: Measure Theory

Provide solutions to 6 of the problems below

I.1 Let f be real-valued.

(a) Give the definition of a measurable function.

(b) If f is measurable, is f^2 measurable? Justify.

(c) If f^2 is measurable, is f measurable? Justify.

I.2 Let X be a countable set and μ a measure on X. Assume that for any $F \subseteq X$ there is $G \subseteq F$ with $\mu(G) < \infty$. Prove the following:

(a) For any $x \in X$, $\mu(\{x\}) < \infty$.

(b) μ is σ-finite.

I.3 Let $(X, \mathcal{M}, \mu)$ be a measure space. Assume $\{f_n\}$ is a sequence in $L^1(X, \mu)$ so that $f_n \to f$ pointwise, and there exists $M > 0$ so that for all n and all $x \in X$ the inequality $|f_n(x)| \leq M$ holds.

(a) If $\mu(X) < \infty$, show that $f \in L^1(X, \mu)$ and $\int f_n \to \int f$.

(b) Show by example that the above conclusion may fail if $\mu(X) = \infty$.

I.4 Let μ^* be an outer measure on a set X. If $E \subseteq X$ satisfies $\mu^*(E) = 0$ prove that E is μ^*-measurable.

I.5 Let $f : [0, 1] \to \mathbb{R}$ be a continuous function of bounded variation such that $f(0) = 0$.

(a) Give an example of such a function for which the identity

$$f(x) = \int_0^x f'(x)dx$$

fails to hold for a.e. $x \in [0, 1]$.

(b) For what type of functions f does the identity in (a) hold almost everywhere?

I.6 (a) State the Monotone Convergence Theorem.

(b) State Fatou's Lemma.

(c) Prove Fatou's Lemma using the Monotone Convergence Theorem or prove the Monotone Convergence Theorem using Fatou's Lemma.

I.7 Define $F : \mathbb{R} \to \mathbb{R}$ by

$$F(x) = \begin{cases} 0 & \text{if } x < 1/4, \\ x & \text{if } 1/4 \leq x < 1, \\ x^2 + 1 & \text{if } 1 \leq x. \end{cases}$$

Let μ_F be the Borel measure associated to F.

(a) Calculate $\mu_F((1/4, 1])$ and $\mu_F([1/4, 1))$.

(b) Calculate the Lebesgue derivative of μ_F.

(c) Give the Lebesgue-Radon-Nikodym decomposition of μ_F with respect to m.

I.8 Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be measures spaces and assume that $E \in \mathcal{M} \times \mathcal{N}$. Show that if $\mu \times \nu(E) = 0$ then $\nu(E_x) = \mu(E^x) = 0$ for a.e. x and y.

Part II: Complex and Functional Analysis

Provide solutions to 3 of the problems below

II.1 Let X be a normed vector space, prove that X^* is a complete normed vector space.

II.2 Assume that $T \in B(\mathcal{H})$ is isometric and surjective where H is a Hilbert space. Prove that T is a unitary operator.

II.3 Let X be a Banach space and $x, y \in X$. If $x \neq y$ prove that there is $\varphi \in X^*$ with $\varphi(x) \neq \varphi(y)$.

II.4 (a) State the Casorati-Weierstrass theorem.

(b) Let f be holomorphic and bounded on the punctured unit disk $\{z \in \mathbb{C} : 0 < |z| < 1\}$. Argue that f has a removable singularity at the origin.

II.5 (a) State the maximum modulus theorem.

(b) Let f be a holomorphic function mapping the unit disk into the unit circle. Show that f is constant.

II.6 Let f be holomorphic on $\mathbb{C}$ and assume that there exists $c > 0$ such that

$$|f(z)| \leq c(1 + \sqrt{|z|})$$

for all complex z. Show that f is constant.