Analysis Preliminary Examination

January 2007

- Unless a problem states otherwise you can assume that μ is an arbitrary measure.

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1. Let m^* denote Lebesgue outer measure on $\mathbb{R}$. Prove that m^* is countably subadditive. Use this to prove that $m^*(\mathbb{Q}) = 0$.

2. Define what it means for a function to be of bounded variation. Give an example of a function which is not of bounded variation, be sure to explain why it is not.

3. For $f : \mathbb{R} \to \mathbb{R}$ show that f is of bounded variation if and only if f is the difference of two bounded increasing functions on $\mathbb{R}$.

4. Let m denote Lebesgue measure on $\mathbb{R}$ and assume $f \in L^1(m)$. Define $F(x) = \int_{(-\infty,x]} f(t)\,dm(t)$ and show that F is continuous on $\mathbb{R}$.

5. (a) Let f and g be μ-measurable functions, and let p and q be conjugate exponents. State Hölder's inequality for f and g.

   (b) If $\mu(X) = 1$, $f \in L^1(\mu)$, and $g \in L^1(\mu)$ with f and g both positive functions satisfying $f(x)g(x) \geq 1 \mu\, a.e.$. Use Hölder's inequality to deduce that

$$\left(\int f\,d\mu\right) \cdot \left(\int g\,d\mu\right) \geq 1.$$

6. Let $(X, \mathcal{M}, \mu)$ be the measure space with $X = \mathbb{N}$, $\mathcal{M} = \mathcal{P}(\mathbb{N})$ and μ equal to counting measure. Define $f : X \times X \to \mathbb{R}$ by

$$f(m,n) = \begin{cases} 1 & m = n \\ -1 & m = n+1 \\ 0 & \text{otherwise} \end{cases}.$$

   Show that f is not $\mu \times \mu$ integrable and further that

$$\int \int f(m,n)\,d\mu(m)\,d\mu(n) \neq \int \int f(m,n)\,d\mu(n)\,d\mu(m).$$

7. If $f : \mathbb{R} \to \mathbb{R}$ is differentiable everywhere and $f'(x)$ is uniformly bounded, show that f is absolutely continuous.

8. If E is a Lebesgue measurable subset of $\mathbb{R}$ with $m(E) > 0$ then for all $0 < \alpha < 1$ there is an open interval I such that $m(E \cap I) > \alpha m(I)$. Here m denotes Lebesgue measure.

9. Let m^* denote Lebesgue outer measure on $[0, 1]$. Define the inner measure $\mu_*(E) = 1 - \mu^*(E^c)$. Show that E is measurable if and only if $\mu^*(E) = \mu_*(E)$.

10. Let $\{f_n\}$ and $\{g_n\}$ be sequences of integrable functions. If there exists f and g such that $f, g \in L^1(\mu)$, $f_n \to f$ μ-a.e., $g_n \to g$ μ-a.e., $|f_n(x)| \leq g_n(x)$ for all x, and $\int g_n\,d\mu \to \int g\,d\mu$ then

$$\lim_{n\to\infty} \int f_n\,d\mu = \int f\,d\mu.$$